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Topics in Advanced Theory I
GAME THEORY

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Repeated Games

Repeated Games with Observable Actions (Perfect Monitoring)

Prisonner's Dilemma

	C	D
C	3,3	0,4
D	4,0	1,1

Let $G = (N, A, u)$ be a finite normal form game. Let $G(T)$ denote the extensive form game where:

- At each stage $t = 1, 2, \ldots, T$, the game G is played among the same players in N
- Players observe the action profiles played in earlier stages,
- Payoffs in G(T) are given by:
  - If $T < \infty$ then $u_i(a_1, \ldots, a_T) = \frac{1}{T}\sum_{t=1}^{T} u_i(a^t)$.
  - If $T = \infty$ then $u_i(a_1, a_2, \ldots) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t)$ for some common discount factor $\delta \in (0, 1)$.
- G is called the **stage game**.

Note: (1) $G(T)$ has perfect recall, it is continuous at infinity, and G is finite so the single-deviation principle applies.

(2) In the discussion of repeated games, we will *restrict attention to pure strategies* in $G(T)$.

(3) We will sometimes write $G^{\delta}(\infty)$ instead of $G(\infty)$.

Minmax, Feasible, & Enforceable Payoffs

The **minmax payoff** of player i is given by:

$$\underline{v}_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

Let (p_{-i}, a_i^v) be a solution of the problem above.

A payoff profile $w \in \mathbb{R}^n$ is **feasible** if it is a convex combination of payoffs $(u(a))_{a \in A}$ in G.

A payoff profile $w \in \mathbb{R}^n$ is **[strictly] enforceable** if $[w_i > \underline{v}_i]$ $w_i \geq \underline{v}_i$ for any $i \in N$. An action profile $a \in A$ is **[strictly] enforceable** if $u(a)$ is [strictly] enforceable.

Example:

	L	M	R
U	2,2	1,1	1,0
M	1,1	4,4	0,6
D	0,1	6,0	0,0

Preliminary Observations

Proposition: Let w be the payoff profile under a NE of $G(T)$. Then w is feasible and enforceable.

Proposition: Suppose that $a(t)$ is a pure strategy NE of G for ant $t = 1, \ldots, T$. Then the strategy profile s in $G(T)$ given by $s(a^1, \ldots, a^t) = a(t+1)$ for any history $(a^1, \ldots, a^t)$ and $t < T$, is an SPE of $G(T)$.

Proposition: Suppose that $T < \infty$ and G has a unique pure strategy NE a^*. Then $G(T)$ has a unique SPE strategy profile s where at each stage players play according to a^*.

Suppose that G has multiple NE and $T < \infty$. Can there be a SPE s of $G(T)$ where $s(h)$ is not a NE of the stage game for some histories h?

A Perfect Folk Theorem with Nash Threats

Theorem: (Friedman, 1971) Suppose that G has a NE a^* and let w be a payoff profile s.t. $w >> u(a^*)$ and $w = u(\bar{a})$ for some $\bar{a} \in A$. Then there is $\bar{\delta} \in (0, 1)$ s.t. for any $\delta \in (\bar{\delta}, 1)$, w is a SPE payoff profile of $G^\delta(\infty)$.

Proof: Let s be s.t. $s(h) = \bar{a}$ if $h = \emptyset$ or $h = (\bar{a}, \bar{a}, \ldots, \bar{a})$. Otherwise let $s(h) = a^*$. Then the payoff under s is $w = u(\bar{a})$. Moreover s is a SPE iff for any i and $a_i \in A$:

$$(1 - \delta)u_i(a_i, \bar{a}_{-i}) + \delta u_i(a^*) \leq u_i(\bar{a})$$

Since $u_i(a^*) < w_i = u_i(\bar{a})$, the above inequality is satisfied for all i and a_i, when δ is large enough. $\square$

Question: *What if w above is a feasible payoff profile that does not correspond to a pure strategy profile $\bar{a}$ of G?*

- Public Randomization
- Time averaging (Fudenberg and Maskin, 1991)

Strategies as Automata

An **automaton** for player i in $G(\infty)$ consists of:

- A set Q_i (the set of **states**)
- An element $q_i^1 \in Q_i$, (the **initial state**)
- A function $f_i \colon Q_i \to A_i$ (the **output function**)
- A function $\tau_i \colon Q_i \times A \to Q_i$ (the **transition function**)

The strategy s_i induced by $(Q_i, q_i^0, f_i, \tau_i)$ is given by:

- For any $h = (a^1, \cdots, a^{t-1})$, find $q_i^t(h)$ inductively:
  - Let $q_i^1(\emptyset) = q_i^1$
  - For $1 < k \leq t-1$, let

$$q_i^k(a^1, \cdots, a^{k-1}) = \tau_i\left(q_i^{k-1}(a^1, \cdots, a^{k-2}), a^{k-1}\right)$$

- Set $s_i(h) = f_i(q_i^t(a^1, \cdots, a^{t-1}))$.

Example: Tit-for-tat.

A More General Perfect Folk Theorem

Theorem: (Fudenberg and Maskin, 1986) Let $\bar{a}$ be a strictly enforceable action profile. Assume that there is a collection $(a(i))_{i \in N}$ of strictly enforceable action profiles s.t. for every i

$$u_i(\bar{a}) \geq u_i(a(i)) \text{ and } u_i(a(j)) > u_i(a(i)) \text{ for } j \neq i.$$

Then there is $\bar{\delta} \in (0,1)$ s.t. for any $\delta \in (\bar{\delta}, 1)$, there is SPE s of $G^{\delta}(\infty)$ in which players play $\bar{a}$ on the equilibrium path.

Note: With public randomization our proof will also imply: *If the set of feasible payoffs has dimension n (**full dimension condition**), then any strictly enforceable feasible payoff profile is an SPE payoff profile of $G^{\delta}(\infty)$ for δ sufficiently close to 1.*

States: $\{C(j) \mid 0 \leq j \leq n\} \cup \{P(j,t) \mid j \in N \;\&\; 1 \leq t \leq L\}$.
Initial state: $C(0)$. (Set $a(0) := \bar{a}$.)
Output function: $a(j)$ in $C(j)$, and (p_{-j}, a_j^v) in $P(j,t)$.
Transitions: If the last play is a:

- From $C(j)$ stay in $C(j)$ unless a single player k deviated from $a(j)$, in which case move to $P(k, L)$.
- From $P(j,t)$:
  - If a single player $k \neq j$ deviated from (p_{-j}, a_j^v) , then move to $P(k, L)$
  - Otherwise go to $P(j, t-1)$ if $t > 1$; to $C(j)$ if $t = 1$.

Let M be the maximum payoff in G. Choose L, δ s.t. $\forall i, j$:

$$M - u_i(a(j)) < L[u_i(a(j)) - v_i]$$

$$M - u_i(a(j)) < \sum_{k=2}^{L+1} \delta^{k-1}[u_i(a(j)) - v_i]$$

$$\sum_{k=1}^{L} \delta^{k-1}[M - u_i(p_{-j}, a_j^v)] < \sum_{k=L+1}^{\infty} \delta^{k-1}[u_i(a(j)) - u_i(a(i))]$$

Other Folk Theorems for $T = \infty$

- Allowing for behavioral strategies. (F&M 1986)

- Abreu, Dutta, and Smith (1994):
  - If no two players have the same vNM preference over A (**NEU**), and if there is public randomization, then every feasible strictly enforceable profile is a SPE payoff profile in $G^\delta(\infty)$ for sufficiently large δ.
  - If there is no $\sigma = (\sigma_1, \ldots \sigma_n)$ that simultaneously gives two players their minimum payoff in G (**NSM**), then NEU is *necessary* for the perfect folk theorem.

- Alternative methods of evaluating payoffs sequences:
  - **Limit of Means** (Aumann and Shapley, 1976) $(v_i^1, v_i^2, \ldots) \succ_i (w_i^1, w_i^2, \ldots) \Leftrightarrow \text{liminf} \frac{1}{T} \sum_{t=1}^{T} (v_i^t - w_i^t) > 0$
  - **Overtaking** (Rubinstein, 1979) $(v_i^1, v_i^2, \ldots) \succ_i (w_i^1, w_i^2, \ldots) \Leftrightarrow \text{liminf} \sum_{t=1}^{T} (v_i^t - w_i^t) > 0$

A Perfect Folk Theorem for Finitely Repeated Games

Theorem: (Krishna, 1989) Let $\bar{a}$ be a strictly enforceable action profile. Assume that (i) for each $i \in N$ there are two NE of G that differ in the payoff of player i and (ii) there is a collection $(a(i))_{i\in N}$ of strictly enforceable action profiles s.t. for every i

$$u_i(\bar{a}) > u_i(a(i)) \text{ and } u_i(a(j)) > u_i(a(i)) \text{ for } j \neq i.$$

Then for any $\epsilon > 0$, there is an integer T^* s.t. if $\infty > T > T^*$ then $G(T)$ has an SPE whose payoffs are within ϵ of $u(\bar{a})$.

Repeated Games with Imperfect Monitoring

Abreu, Pearce, & Stachetti (1990)

The Stage Game G

There is a finite set of players N, each player i has finite set of actions A_i.

Publicly observable **signals** $p \in \Omega$. (Ω is finite)

$f(p \,|\, a)$: probability p conditional on the action profile a.

Realized payoffs: $\pi_i(a_i, p)$

Expected payoffs: $\Pi_i(a) = \sum_{p \in \Omega} f(p|a)\pi_i(a_i, p)$.

Player i's expected payoffs depends on the others' actions only through the distribution of the public signal.

Important Assumptions:

- $f(\cdot \,|\, a)$ has full support for any $a \in A$.
- The stage Game G has a pure strategy NE.

The Infinitely Repeated Game

We will restrict attention to *pure strategies*. A **strategy** of player i is a sequence $(\sigma_i(1), \sigma_i(2), \ldots)$ where:

$$\sigma_i(t)\colon \Omega^{t-1} \times A_i^{t-1} \to A_i \qquad \text{for } t \geq 1.$$

$v(\sigma)$ = the **expected average discounted payoff** from the strategy profile σ.

Equilibrium payoffs:

$V = \{v(\sigma) \mid \sigma$ is a sequential eqm. of the repeated game$\}$.

Note:

- The one deviation property holds for the repeated game.
- V is a nonempty, and bounded subset of $\mathbb{R}^N$.

A Section from the Repeated Game: The Augmented Game

Given a function $u\colon \Omega \to \mathbb{R}^N$ (promised utilities), let

$$E(a,u) = (1-\delta)\Pi(a) + \delta \sum_{p \in \Omega} f(p \,|\, a)\, u(p).$$

Defn: A pair (a,u) where $a \in A$ and $u\colon \Omega \to \mathbb{R}^N$ is **admissible w.r.t.** $W \subset \mathbb{R}^N$ if:

1. $u(\Omega) \subset W$, and
2. $E_i(a,u) \geq E_i(a'_i, a_{-i}; u)$ for any $i \in N$ and $a'_i \in A_i$.

Defn: $B(W) = \{E(a,u) \mid (a,u)$ is admissible w.r.t. $W\}$.

Defn: W is **self-generating** if $W \subset B(W)$.

Results that We Will Prove

Theorem 1 *(Self-Generation) If W is bounded and self-generating then $B(W) \subset V$. In fact $B^k(W) \subset V$.*

Note: Boundedness of W is a necessary assumption.

Theorem 2 *(Factorization) $V = B(V)$. Hence V is the largest bounded fixed point of B.*

Theorem 6 *(Monotonicity) $\delta_2 > \delta_1 \Rightarrow V^1 \subset V^2$. [Finiteness of Ω requires that we extend the model to allow for public randomization to prove this result.]*

Vector Notation: $p^t = (p_1, \ldots, p_t)$.

For any strategy profile σ and history $h = (p^{t-1}, a^{t-1}) \in \Omega^{t-1} \times A^{t-1}$, let $\sigma|_h$ denote the **continuation strategy profile conditional on** h, i.e.:

$$\sigma_i|_h(\tau)(\tilde{p}^{\tau-1}, \tilde{a}_i^{\tau-1}) = \sigma_i(t+\tau-1)\left(\left(p^{t-1}, \tilde{p}^{\tau-1}\right), \left(a_i^{t-1}, \tilde{a}_i^{\tau-1}\right)\right).$$

Proof of Self-Generation

For any $x \in B(W)$ let $(a(x), U(x))$ be an admissible pair w.r.t. W s.t. $x = E(a(x), U(x))$.

let $w \in B(W)$. Define the functions $U^t(w) \colon \Omega^t \to \mathbb{R}^N$ and the strategy profile $\sigma(w)$ recursively:

$$U^1(w)(p) = U(w)(p) \quad \text{and} \quad \sigma(w)(1) = a(w)$$

$$U^{t+1}(w)(p^{t+1}) = U\left(U^t(w)(p^t)\right)(p_{t+1})$$

$$\sigma(w)(t+1)(p^t) = a\left(U^t(w)(p^t)\right)$$

It remains to verify that:

- $v(\sigma(w)) = w$, and
- No agent has a profitable single-deviation under $\sigma(w)$.

Proof of Self-Generation (continued): $v\left(\sigma(w)\right) = w$

Let $x \in B(W)$. Then:

$$\begin{aligned} v(\sigma(x)) &= (1-\delta)\Pi(a(x)) + \delta \sum_{p \in \Omega} f(p \,|\, a) v\left(\sigma\left(U(x)(p)\right)\right) \\ x &= (1-\delta)\Pi(a(x)) + \delta \sum_{p \in \Omega} f(p \,|\, a) U(x)(p) \end{aligned}$$

implying:

$$v(\sigma(x)) - x = \delta \sum_{p \in \Omega} f(p \,|\, a) \left[v\left(\sigma\left(U(x)(p)\right)\right) - U(x)(p) \right]$$

$$\Rightarrow |v(\sigma(x)) - x| \leq \delta \sup_{y \in W} |v\left(\sigma\left(y\right)\right) - y| \leq \delta \sup_{y \in B(W)} |v\left(\sigma\left(y\right)\right) - y|$$

Taking sup on the l.h.s.:

$$\sup_{x \in B(W)} |v(\sigma(x)) - x| \leq \delta \sup_{y \in B(W)} |v\left(\sigma\left(y\right)\right) - y|$$

Since the supremum is finite $v(\sigma(x)) = x$ for any $x \in B(W)$.

Proof of Self-Generation (continued): $\sigma(w)$ is a SE.

To verify that there there are no profitable single deviations, take any history p^t of public signals. Let $x = U^t(w)(p^t) \in W$. Since $(a(x), U(x))$ is admissible w.r.t. W:

$$(1-\delta)\Pi_i(a_i(x), a_{-i}(x)) + \delta \sum_{p\in\Omega} f(p \,|\, a(x)) U_i(x)(p)$$

$$\geq (1-\delta)\Pi_i(a_i', a_{-i}(x)) + \delta \sum_{p\in\Omega} f(p \,|\, a_i', a_{-i}(x)) U_i(x)(p)$$

Plug in

$a(x) = \sigma(w)(t+1)(p^t)$,

$\sigma(w)|_{p^t,p} = \sigma\left(U^{t+1}(w)(p^t,p)\right) = \sigma\left(U(x)(p)\right)$, and

$v\left(\sigma\left(U(x)(p)\right)\right) = U(x)(p)$.

Hence i has no profitable deviations after p^t. $\square$

Proof of Factorization: $V = B(V)$

By Self-Generation, we only need to show $V \subset B(V)$. Let $w = v(\sigma) \in V$ where σ is a SE. We need to find a pair (a, u) admissible w.r.t V s.t. $w = E(a, u)$.

Set $a := \sigma(1)$ and $u(p) := v(\sigma|_{p,a})$ for any $p \in \Omega$. Then $w = v(\sigma) = E(a, u)$

$\sigma|_{p,a}$ is an SE (Why?). Hence $u(p) \in V$.

Consider a deviation by player i where he plays a_i' at $t = 1$ and plays according to σ_i subsequently. Since σ is a SE, this is not profitable, i.e.:

$$(1-\delta)\Pi_i(a_i', a_{-i}) + \delta \sum_{p \in \Omega} f(p \,|\, a_i', a_{-i}) v_i\,(\sigma|_{p,a})$$

$$\leq (1-\delta)\Pi_i(a) + \delta \sum_{p \in \Omega} f(p \,|\, a) v_i\,(\sigma|_{p,a})$$

Therefore $E_i(a_i', a_{-i}, u) \leq E_i(a, u)$. □

Proof of Monotonicity

$B^2\left(V^1\right) = B^2\left(co\left(V^1\right)\right)$ by public randomization.

Let $\delta_2 > \delta_1$. By Thm 1, we only need to show V^1 is a self-generating set at δ_2. Let $w \in V^1$. Then at δ^1, there is an admissible pair (a, u^1) w.r.t V^1 s.t. $w = E^1(a, u^1)$, i.e.:

$$w = (1 - \delta_1)\Pi(a) + \delta_1 \sum_{p \in \Omega} f(p\,|\,a)u^1(p)$$

$$\begin{aligned}
\Leftrightarrow w - \Pi(a) &= \frac{\delta_1}{1-\delta_1} \sum_{p \in \Omega} f(p\,|\,a)\left[u^1(p) - w\right] \\
&= \frac{\delta_2}{1-\delta_2} \sum_{p \in \Omega} f(p\,|\,a)\kappa\left[u^1(p) - w\right].
\end{aligned}$$

where $\kappa := [\delta_1/(1-\delta_1)]/[\delta_2/(1-\delta_2)] \in (0, 1)$.

$u^2(p) - w := \kappa[u^1(p) - w] \Rightarrow u^2(p) = \kappa u^1(p) + (1-\kappa)w \in co(V^1)$.

It remains to verify incentives for (a, u^2) at δ_2.

Proof of Monotonicity (continued) Verifying Incentives for (a, u^2) at δ_2

Incentives for (a, u^1) at δ_1:

$$\Pi_i(a_i', a_{-i}) - \Pi_i(a) \;\leq\; \frac{\delta_1}{1-\delta_1} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i}) \Big] \, u_i^1(p)$$

This is equivalent to:

$$\Pi_i(a_i', a_{-i}) - \Pi_i(a) \;\leq\; \frac{\delta_1}{1-\delta_1} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i}) \Big] \Big[u_i^1(p) - w_i \Big]$$

$$\begin{aligned} \text{r.h.s} \;&=\; \frac{\delta_2}{1-\delta_2} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i}) \Big] \, \kappa \Big[u_i^1(p) - w_i \Big] \\ &=\; \frac{\delta_2}{1-\delta_2} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i}) \Big] \Big[u_i^2(p) - w_i \Big] \end{aligned}$$

Hence the incentives for (a, u^2) at δ_2 are satisfied. $\square$

Technical Results

Theorems 3, 7 and the Corollary are invalid in our finite Ω model, they require a continuous signal distribution.

Theorem 3 *(Sufficiency of Bang-Bang Reward Functions) Let W be compact. Then for any (a, u) that is admissible w.r.t. $co(W)$, there is u' s.t. (i) (a, u') is admissible w.r.t. $ext(W)$ and (ii) $E(a, u') = E(a, u)$.*

Corollary For any compact W, $B(W) = B\left(co(W)\right)$.

Theorem 4 *V is compact.*

Theorem 5 *(Algorithm) Let W be a compact set such that $V \subset B(W) \subset W$. Then $B^k(W) \searrow V$.*

"Theorem 7" *(Necessity of Bang-Bang Reward Functions) Under certain conditions, the reward functions faced by players in efficient equilibria* **must take** *values in $ext(V)$.*