# Repeated Games

## *Long-Run versus Short-Run Player*

a fixed simultaneous move *stage game*

Player 1 is long-run with discount factor $\delta$

actions $a^1 \in A^1$ a finite set

utility $u^1(a^1, a^2)$

Player 2 is short-run with discount factor 0

actions $a^2 \in A^2$ a finite set

utility $u^2(a^1, a^2)$

the "short-run" player may be viewed as a kind of "representative" of many "small" long-run players

## *Repeated Game*

history $h_t = (a_1, a_2, \ldots, a_t)$

null history $h_0$

behavior strategies $\alpha_t^i = \sigma^i(h_{t-1})$

*Equilibrium*

Nash: usual definition

Subgame perfect: usual definition, Nash after each history

Observation: the repeated static equilibrium of the stage game is a subgame perfect equilibrium of the finitely or infinitely repeated game

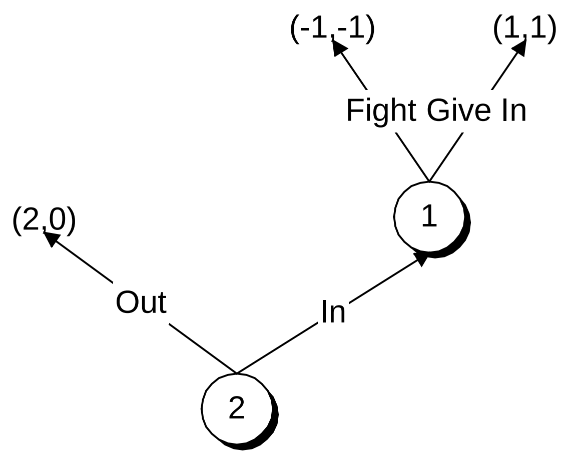
strategies: play the static equilibrium strategy no matter what

“perfect equilibrium with public randomization”

may use a public randomization device at the beginning of each period to pick an equilibrium

key implication: set of equilibrium payoffs is convex

## *Example: chain store game*




normal form

| | out | in |
|---|---|---|
| fight | 2,0* | -1,-1 |
| give in | 2,0 | 1,1** |

Nash

subgame perfect is In,Give In

variation on chain store

| | out | in |
|---|---|---|
| fight | 2-$\varepsilon$, 0 | -1,-1 |
| give in | 2,0 | 1,1** |

now the only equilibrium  is In, Give In payoff at static Nash equilibrium to LR player: 1

precommitment or Stackelberg equilibrium

precommit to fight get $2-\varepsilon$

minmax payoff to LR player: 1 by giving in

utility to long-run player

precommitment/Stackelberg = $2-\varepsilon$

best dynamic equilibrium = ?

Set of dynamic equilibria

static Nash = 1

worst dynamic equilibrium = ?

minmax = 1

## *Repeated Chain Store*

finitely repeated game

final period: In, Give, so in every period

Do you believe this??

*Infinitely repeated game*

begin by playing Out, Fight

if Fight has been played in every previous period then play Out, Fight

if Fight was not played in a previous period play In, Give In (reversion to static Nash)

claim: this is subgame perfect

clearly a Nash equilibrium following a history with Give In

SR play is clearly optimal

for LR player

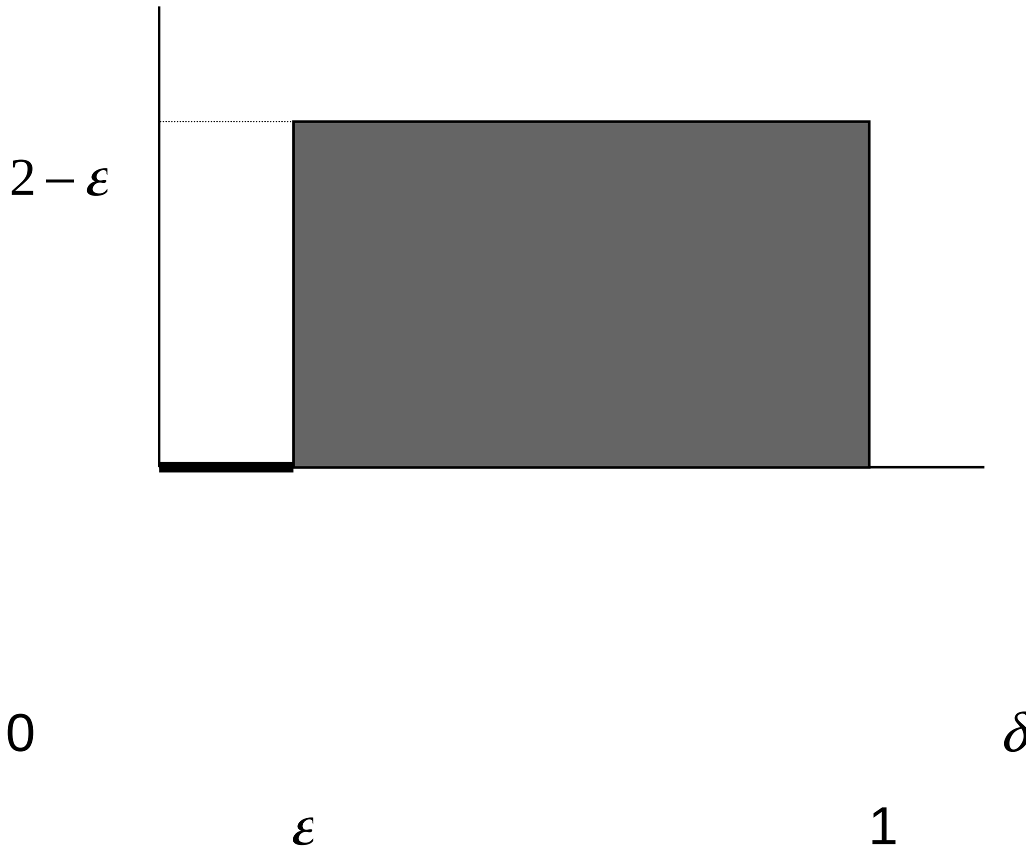
may Fight and get $2-\varepsilon$

or give in and get $(1-\delta)2+\delta 1$

so condition for subgame perfection

$$
\begin{aligned}
&2-\varepsilon \geq (1-\delta)2+\delta 1 \\
&\delta \geq \varepsilon
\end{aligned}
$$

equilibrium utility for LR



*General Deterministic Case*

Fudenberg, Kreps and Maskin [1990]

utility to long-run player

max $u^1(a)$

mixed precommitment/Stackelberg

pure precommitment/Stackelberg

$\bar{v}^1$ best dynamic equilibrium

Set of dynamic equilibria

static Nash

$\underline{v}^1$ worst dynamic equilibrium

minmax

min $u^1(a)$

*Characterization of Equilibrium Payoff*

$\alpha = (\alpha^1, \alpha^2)$ where $\alpha^2$ is a b.r. to $\alpha^1$

$\alpha$ represent play in the first period of the equilibrium

$w^1(a^1)$ represents the equilibrium payoff beginning in the next period

$$v^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$v^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$\underline{v}^1 \leq w^1(a^1) \leq \overline{v}^1$$

## *Characterization of Best/Worst Equilibrium Payoffs*

maximize $\bar{v}^1$, minimize $\underline{v}^1$ subject to

$\alpha = (\alpha^1, \alpha^2)$ where $\alpha^2$ is a b.r. to $\alpha^1$

$$\bar{v}^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$\bar{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$\underline{v}^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$\underline{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$\underline{v}^1 \leq w^1(a^1) \leq \bar{v}^1$$

## *Remarks*

1) problem simplifies if static Nash = minmax

2) if $v^1 \geq (1-\delta)u^1(a^1,\alpha^2)+\delta w^1(a^1)$ then $v^1 \geq (1-\delta)u^1(a^1,\alpha^2)+\delta \underline{v}^1$

simplification: split into two problems by defining $n^1$ as static Nash payoff

$$n^1 \le w^1(a^1) \le \bar{v}^1$$

$$\underline{v}^1 \le w^1(a^1) \le n^1$$

as $\delta \to 1$ $w^1(a^1) \to \bar{v}^1, \underline{v}^1$ in the two problems so this is OK *max problem*

fix $\alpha = (\alpha^1, \alpha^2)$ where $\alpha^2$ is a b.r. to $\alpha^1$

$$\bar{v}^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$\bar{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0$$

$$n^1 \leq w^1(a^1) \leq \bar{v}^1$$

how big can $w^1(a^1)$ be in = case?

Biggest when $u^1(a^1, \alpha^1)$ is smallest, in which case

$$w^1(a^1) = \bar{v}^1$$

$$\bar{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta \bar{v}^1$$

conclusion for fixed $\alpha$

$\min_{a^1|\alpha(a^1)>0} u^1(a^1,\alpha^2)$

i.e. worst in support

$\bar{v}^1 = \max_{\alpha^2 \in BR^2(\alpha^1)} \min_{a^1|\alpha(a^1)>0} u^1(a^1,\alpha^2)$

observe:

mixed precommitment $\geq \bar{v}^1 \geq$ pure precommitment

Modified Chain Store Example

| | out | in |
|---|---|---|
| fight | 2-$\varepsilon$, 0 | -1,-1 |
| give in | 2,0 | 1,1 |

| *p*(fight) | BR | worst in support |
|---|---|---|
| 1 | out | $2-\varepsilon$ |
| ½<*p*<1 | out | $2-\varepsilon$ |
| 0<*p*<½ | in | -1 |
| p=0 | in | 1 |

check: $w^1(a^1) = \dfrac{\overline{v}^1 - (1-\delta)u^1(a^1, \alpha^2)}{\delta} \geq n^1$

as $\delta \to 1$ then $w^1(a^1) \to \overline{v}^1 \geq n^1$

*min problem*

fix $\alpha = (\alpha^1, \alpha^2)$ where $\alpha^2$ is a b.r. to $\alpha^1$

$$\underline{v}^1 \geq (1-\delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)$$

$$\underline{v}^1 \leq w^1(a^1) \leq n^1$$

Biggest $u^1(a^1, \alpha^1)$ must have smallest $w^1(a^1) = \underline{v}^1$

$$\underline{v}^1 = (1-\delta)u^1(a^1, \alpha^2) + \delta \underline{v}^1$$

conclusion

$$\underline{v}^1 = \max u^1(a^1, \alpha^2)$$

or

$$\underline{v}^1 = \min_{\alpha^2 \in BR^2(\alpha^1)} \max u^1(a^1, \alpha^2)$$

that is, constrained minmax

## *Sample Calculation*

| | L | M | R |
|---|---|---|---|
| U | 0,-3 | 1,2 | 0,3 |
| D | 0,3* | 2,2 | 0,0 |

static Nash gives 0

minmax gives 0

worst payoff in fact is 0

pure precommitment also 0

## *Mixed Precommitment*

$p$ is probability of up

to get more than 0 must get SR to play M

$-3p+(1-p)3 \le 2$ and $3p \le 2$

first one

$-3p+(1-p)3 \le 2$

$-3p-3p \le -1$

$p \ge 1/6$

second one

$3p \le 2$

$p \le 2/3$

want to play D so take $p = 1/6$

get $1/6 + 10/6 = 11/6$

utility to long-run player

max $u^1(a)$=2

mixed precommitment/Stackelberg=11/16

$\bar{v}^1$ best dynamic equilibrium=1

pure precommitment/Stackelberg=0

Set of dynamic equilibria

static Nash=0

$\underline{v}^1$ worst dynamic equilibrium=0

minmax=0

min $u^1(a)$=0

## *Calculation of best dynamic equilibrium payoff*

$p$ is probability of up

| $p$ | $BR^2$ | worst in support |
|---|---|---|
| <1/6 | L | 0 |
| 1/6<*p<5/6* | M | 1 |
| p>5/6 | R | 0 |

so best dynamic payoff is 1

## *Moral Hazard*

choose $a^i \in A$

observe $y \in Y$

$\rho(y|a)$ probability of outcome given action profile

private history: $h^i = (a_1^i, a_2^i, \ldots)$

public history: $h = (y_1, y_2, \ldots)$

strategy $\sigma^i(h^i, h) \in \Delta(A^i)$

"public strategies"

*perfect public equilibrium*

## *Moral Hazard Example*

mechanism design problem

each player is endowed with one unit of income

players independently draw marginal utilities of income $\eta \in \{\overline{\eta}, \underline{\eta}\}$

player 2 (SR) has observed marginal utility of income

player 1 (LR) has unobserved marginal utility of income

player 2 decides whether or not to participate in an insurance scheme player 1 must either announce his true marginal utility or he may announce $\overline{\eta}$ independent of his true marginal utility

non-participation: both players get $\gamma = \frac{\overline{\eta} + \underline{\eta}}{2}$

participation: the player with the higher marginal utility of income gets both units of income normal form

| | non-participation | participate |
|---|---|---|
| truth | $\gamma, \gamma$ | $\frac{\overline{\eta}+\gamma}{2}, \frac{\overline{\eta}+\gamma}{2}$ |
| lie | $\gamma, \gamma$ | $\frac{3\gamma}{2}, \frac{\overline{\eta}}{2}$ |

$p^* = \frac{\eta}{\gamma}$ makes player 2 indifferent max $u^1(a) = \frac{3\gamma}{2}$

mixed precommitment/Stackelberg= $\frac{\overline{\eta}+\gamma}{2} + (1-\frac{\underline{\eta}}{\gamma})\frac{\underline{\eta}}{2}$

$\overline{v}^1$ best dynamic equilibrium= $\frac{\overline{\eta}+\gamma}{2}$

pure precommitment/Stackelberg= $\frac{\overline{\eta}+\gamma}{2}$

Set of dynamic equilibria

static Nash= $\gamma$

$\underline{v}^1$ worst dynamic equilibrium= $\gamma$

minmax= $\gamma$

min $u^1(a) = \gamma$

## *Solving the moral example*

player 1 plays "truth" with probability $p*$ or greater

player 2 plays "participate"

$$\bar{v} = (1-\delta)\frac{\bar{\eta}+\gamma}{2} + \delta\left(\frac{1}{2}w(\underline{\eta}) + \frac{1}{2}w(\bar{\eta})\right)$$

$$\bar{v} \geq (1-\delta)\frac{3\gamma}{2} + \delta w(\bar{\eta})$$

$$\bar{v} \geq w(\underline{\eta}), w(\bar{\eta})$$

$w(\bar{\eta})$ must be as large as possible, so inequality must bind; $w(\underline{\eta}) = \bar{v}$

$$\bar{v} = (1-\delta)\frac{3\gamma}{2} + \delta w(\bar{\eta})$$

solve two equations

$$\bar{v} = \bar{\eta} - \frac{\gamma}{2}$$

$$w(\bar{\eta}) = \frac{\bar{v} - (1-\delta)3\gamma / 2}{\delta}$$

check that $w(\overline{\eta}) \geq \gamma$

leads to $\delta \geq 2\left(2 - \frac{\overline{\eta}}{\gamma}\right)$

from $\delta < 1$ this implies

$\overline{\eta} > 3\underline{\eta}$

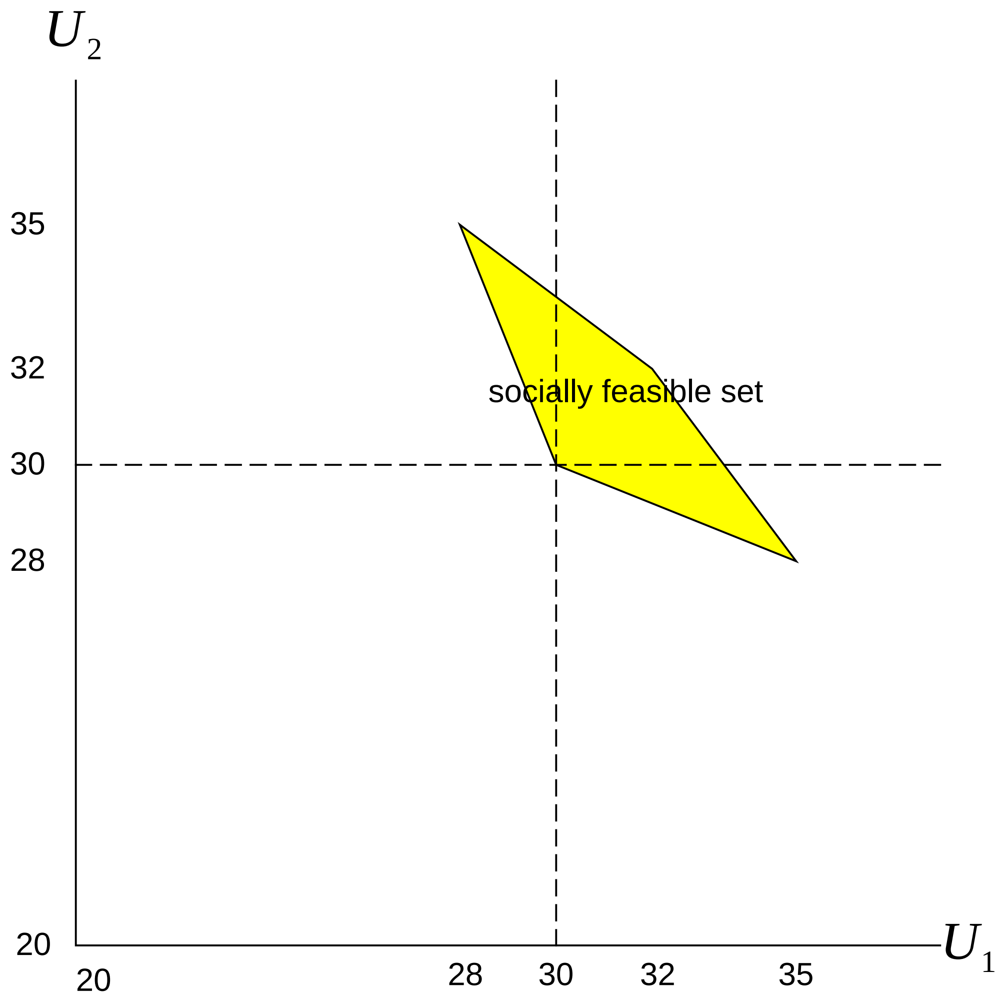
## *Long-Run Players and the Folk Theorem*

## *Folk Theorems*

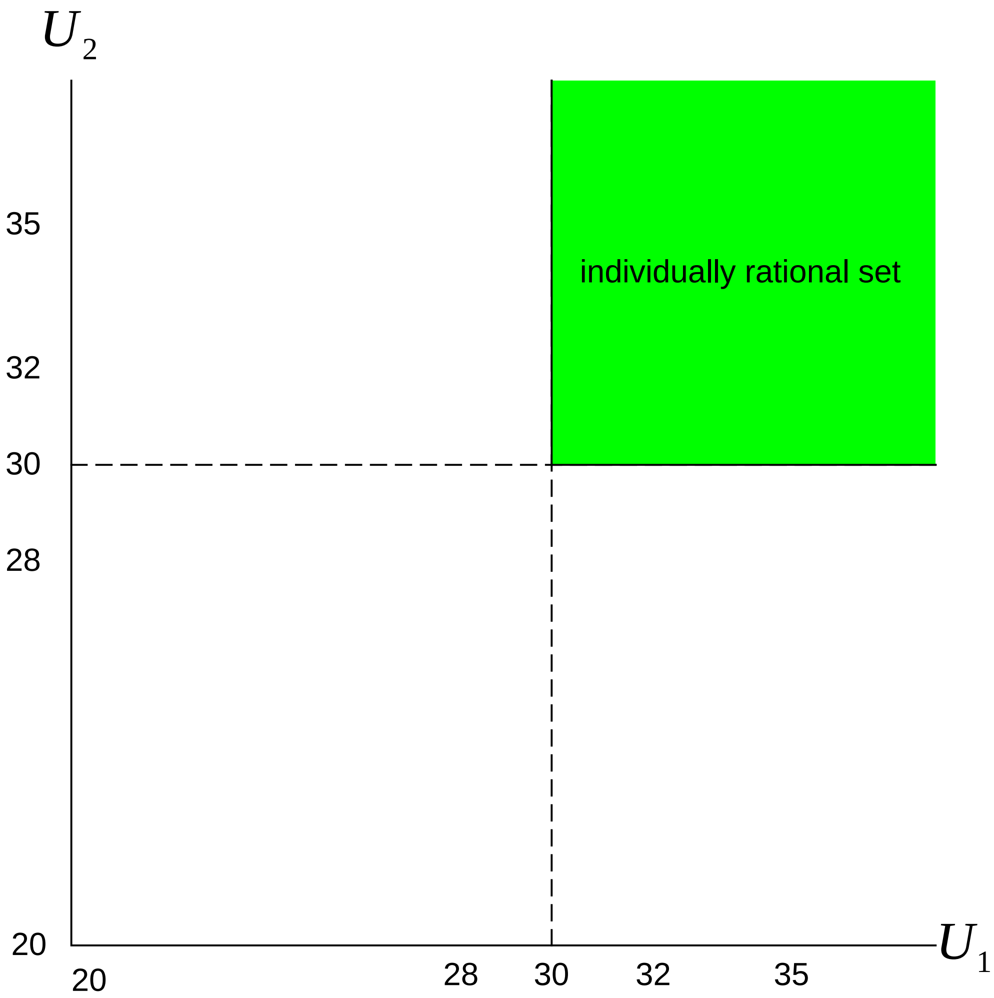
- socially feasible
- individually rational

Statement of Folk Theorem

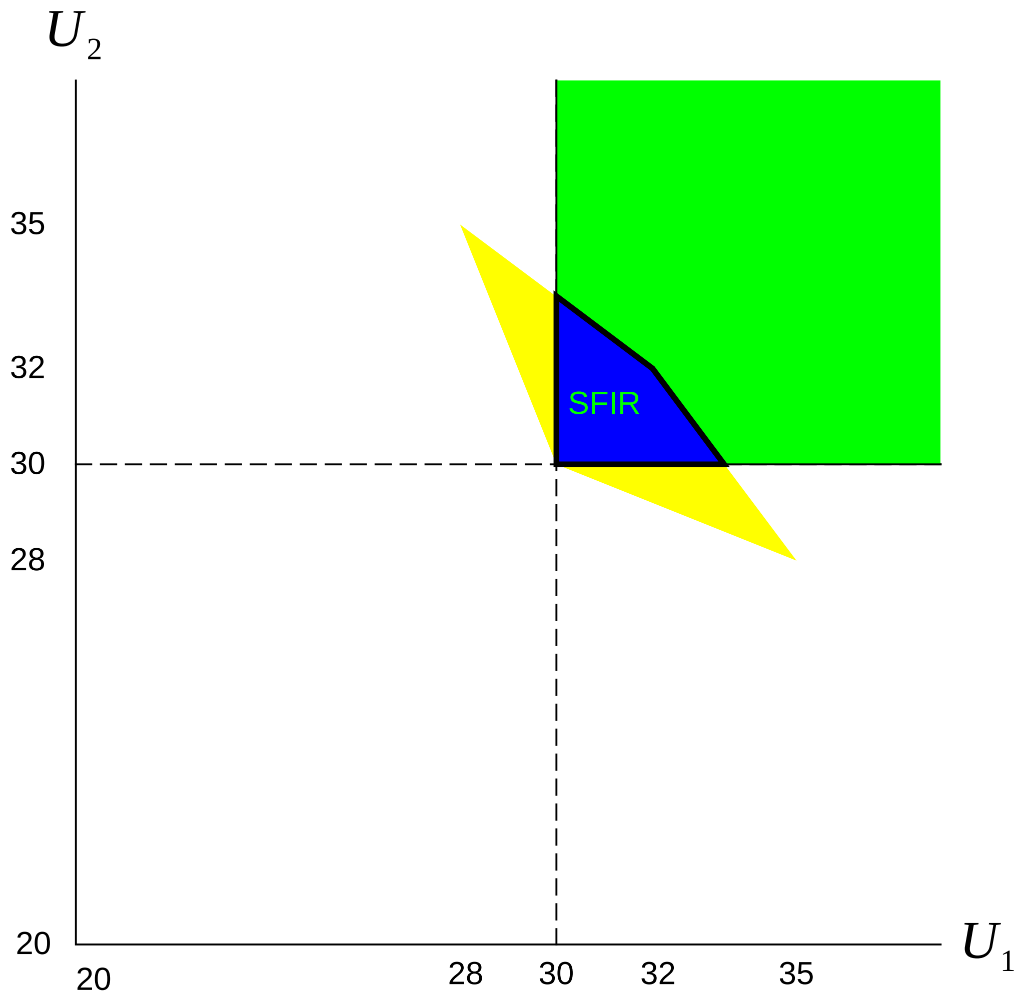
| | Player 2 | |
|---|---|---|
| Player 1 | don’t confess | confess |
| don’t confess | 32,32 | 28,35 |
| confess | 35,28 | 30,30 |



$U_2$
35
32
30
28
20
socially feasible set
$U_1$
20
28
30
32
35



$U_2$
35
32
30
28
20
individually rational set
$U_1$
20
28
30
32
35


$U_2$
35
32
30
28
20
SFIR
$U_1$
20
28
30
32
35

- Nash with time averaging
- perfect Nash threats with discounting
- Fudenberg and Maskin [1986]

## *The Downside of the Folk Theorem*

| | |
|---|---|
| 4,4 | 1,1 |
| 1,1 | 0,0 |

$\delta = 3/4$

D in first period

If DD in first period UU forever after

Else start over

In equilibrium get $(1/4)0 + (3/4)4 = 3$

Deviation get $(1/4)1 + (3/4)3 = 10/4 = 2.5$

In general want $(1-\delta)0+\delta 4=\delta 4 \geq (1-\delta)1+\delta^2 4$

Or

$$0 \geq \delta^2 4 - 5\delta + 1$$

$$\delta = \frac{5 \pm \sqrt{25-4}}{2} = \frac{5-\sqrt{21}}{2} \approx .2087$$

For $\delta$ close to 1 the worst equilibrium is near $1$ for both players

*Tit-for-tat*

Play the same thing that your opponent did in the previous period, cooperate in the first period

| | |
|---|---|
| 3,3 | 0,4 |
| 4,0 | 1,1 |

If your opponent is playing tit-for-tat, use dynamic programming

Four markov strategies:

Do the same as opponent: 3

Do opposite of opponent: $\frac{1-\delta}{1-\delta^2}4 = \frac{4}{1+\delta}$ (=3 at $\delta = 1/3$)

Always cooperate: 3

Always cheat: $(1-\delta)4 + \delta 1 = 4 - 3\delta$ (=3 at $\delta = 1/3$)

So tit-for-tat an equilibrium for $\delta \geq 1/3$

## *Matching and Information Systems*

Juvenal in the first century A.D.

“Sed quis custodiet ipsos custodes?”

translation: “Who shall guard the guardians?”

answer: they shall guard each other.

*Contagion Equilibrium*

players randomly matched in a population; observe only opponent’s current play

Ellison [1993]: could have cooperation due to contagion effects

| | |
|---|---|
| 3,3 | 0,4 |
| 4,0 | 1,1 |

Strategy: cooperate as long as everyone you have ever met cooperated; if you have ever met a cheater, then cheat

With these strategies the number of cheaters is a Markov chain with two aborbing states: all cheat, none cheat

Playing the proposed equilibrium strategy results in non cheat and a utility of 3; deviating results eventually in all cheat; this aborbing state is approached exponentially fast; the amount of time depends on the population size, but not the discount factor, so for discount factor close enough to one it is optimal not to cheat

But contagion effects diminish as population size grows, and the equilibrium is not robust to noise, which will trigger a collapse

## *Information Systems-Example*

Overlapping generations; young matched against old:

Only the young have a move – give a gift to old person

Gift worth $x > 1$ to old person; costs 1 to give the gift

Information system: assigns a young person a flag based on their action and the old person's flag

Consider the following information system and strategies:

Cooperate against a green flag -> green flag

Cheat against a red flag -> green flag

On the other hand

Cheat against green flag -> red flag

Cooperate against red flag -> red flag

If you meet a green flag:

Cooperate you get $x - 1$

Cheat you get $0$

If you meet a red flag

Cheat you get $x$

Cooperate you get $-1$

So it is in fact optimal to cooperate against green (your team) and cheat against red (the other team)

Notice that this is a **strict** Nash equilibrium if there is noise (so that there are some red flags)

Notice that always cheat no matter what the flags is also a strict Nash equilibrium

*Information Systems-Folk Theorem*

Kandori [1992]

$u^i(a)$

$I$ a finite set of information states

$\eta: A \times I^2 \to I$ an information system

if at $t$ you and your opponent played $a_t$ and had states $\eta_t^i, \eta_t^{-i}$, then your next state is $\eta_{t+1}^i = \eta(a_t, \eta_t^i, \eta_t^{-i})$

players randomly matched in a population

observe their current opponents current state

Folk Theorem for information systems: socially feasible individually rational payoff – exists an information system that supports it

*Example*

Prisoner's dilemma

| | C | D |
|---|---|---|
| C | $x, x$ | $0, x+1$ |
| D | $x+1, 0$ | 1,1 |

$I = \{r, g\}$

$$\eta(a^i, \eta^{-i}) = \begin{cases} G & (a^i, \eta^{-i}) = C, G \\ R & (a^i, \eta^{-i}) = C, R \\ R & (a^i, \eta^{-i}) = D, G \\ G & (a^i, \eta^{-i}) = D, R \end{cases}$$

"green team strategy"

defect on red

cooperate on green

$$V(g) = x$$
$$V(r) = \delta x$$

C $(1-\delta)x + \delta V(g) = x$

D $(1-\delta)(x+1) + \delta V(r) = (1-\delta)(x+1) + \delta^2 x =$
$(1-\delta) + (1-\delta+\delta^2)x$

$$x \geq (1-\delta) + (1-\delta+\delta^2)x$$

So $\delta(1-\delta)x \geq (1-\delta)$

$$\delta \geq 1/x$$