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Private Information and the Problem of Coordinating Punishments

Repeated game equilibria have a self-referential nature: players don’t do things because they are afraid they will be punished, and they punish because they are afraid if they do not they will be punished for that and so forth.

- This requires players to know when they are being punished.
- This is difficult with signals that are not common knowledge.
- Is my price low because you deviated or because you got a signal that you should punish me? In the former case I should punish you, in the latter case if I do I trigger off a war that unravels the equilibrium.
Stage Game

two players \( i = 1, 2 \)
chooses an actions \( a_i \) from a finite set \( A_i \)
payoff to an action profile \( g_i(a) \)
\( \bar{g} = \max_{i,a} |g_i(a)| \)
each player observes a private signal \( z_i \) in a finite set \( Z_i \)
action profiles induce a probability distribution \( \pi_a \) over outcomes \( z \)
end of each stage of the game, players make announcements \( y_i \in Y^* \),
where \( Y^* \) is a finite set that is the same for each player
stage game strategy \( s_i = (a_i, m_i) \): choice of action \( a_i \) and map \( m_i : Z_i \rightarrow Y^* \) from private signal to announcements
Remark on the Two Player Case

more players is easier: can compare announcement by different players
Repeated Game

each period $t = 1, 2, \ldots$, stage game is played

public randomization device each period uniform $w \in [0, 1]$

public history at time $t$, $h(t)$: announcements and realization of $w$
signals in all previous periods, and also the realization of $w$ in period $t$, so

$$h(t) = (w(1), y(1), w(2), y(2), \ldots, w(t - 1), y(t - 1), w(t)).$$

private history for player $i$ at time $t$ is

$$h_i(t) = (a_i(1), z_i(1), a_i(2), z_i(2), \ldots, a_i(t - 1), z_i(t - 1)).$$
Strategies

strategy for player $i$ is a sequence of maps $\sigma_i(t)$ mapping the public and private histories $h(t), h_i(t)$ to probability distributions over $S_i$

partial strategy is the strategy conditional on the initial realization of the public randomization device

public strategy is a strategy that depends only on $h(t)$

null private history for player $i$ is $h_i(1)$

initial public history is $h(1)$

for each public history $h(t)$ the public strategy profile $\sigma$ induces partial strategy profile over the repeated game beginning at $t$; denote by $[\sigma \mid h(t)]$
Preferences

discount factor by \( \delta \), use average present value

given strategy profile \( \sigma \) expected average present value of payoffs

generated by partial strategy profiles \([\sigma \mid w(1)]\) denoted by \( G_i(\sigma, \delta) \)

perfect public equilibrium a public strategy profile \( \sigma \) such that for any

public history \( h(t) \) and any private partial strategy \( \tilde{\sigma}_i \) by any player \( i \) we have

\[
G_i([\sigma \mid h(t)], \delta) \geq G_i((\tilde{\sigma}_i, [\sigma_{-i} \mid h(t)]), \delta).
\]

by standard dynamic programming arguments sufficient to consider

deviations to public strategies.
Structure of Information

convenient to think of players “agreeing” if they make same announcement as each other

think of \( Y^* \) as being the subset of \( Y \) in which \( y_1 = y_2 \): called diagonal

given message profile \( m \) the information structure \( \pi \) induces a distribution over the diagonal of announcement profiles

probability of diagonal point \( \pi_a^m(y^*) = \sum_{z|m_1(z_1) = m_2(z_2) = y^*} \pi_a(z) \),

probability of joint announcement conditional on diagonal

\[
\pi_a^m(y^* \mid Y^*) = \frac{\pi_a^m(y^*)}{\sum_{y \in Y^*} \pi_a^m(y)}
\]

probability opponent’s message given positive probability signal

\[
\pi_a^m(y_{-i} \mid z_i) = \sum_{z_{-i}|m_{-i}(z_{-i}) = y_{-i}} \pi_a(z_{-i} \mid z_i)
\]
Almost Public Messaging

Definition 1: A game has \((\varepsilon, \nu)\) public information with respect to \(m\) if for all action profiles \(a\),

\[
(1) \quad \bar{\pi}_a^m \equiv \sum_{y^* \in Y^*} \pi_a^m(y^*) \geq 1 - \varepsilon
\]

\[
(2) \quad \text{if } \pi_a(z) > 0 \text{ then for all } y_{-i} \neq m_i(z_i), \\
\pi_a^m(y_{-i} \mid z_i) \leq \pi_a^m(m_i(z_i) \mid z_i) - \nu
\]

most of the time, each player fairly confident of the other player's message

limit case of \((0, \nu)\)-public information two players’ messages are perfectly correlated, so public information
Versus “Close to Public Monitoring”

similar to the Mailath and Morris “ε-close to public monitoring” but weaker in two ways

1. Mailath and Morris suppose each players private signal \( z_i \) lie in same set as the signals in the limiting public-information game meaning
\[
\#Y^* = \#Z_i
\]

2. they suppose that in public information limit, every signal has strictly positive probability under every action profile, and that the distribution of each player's private signals is close to this limit

these imply condition (1) a stronger version of condition (2):
\[
\lim_{\varepsilon \to 0} \pi_m^a(m_i(z_i) \mid z_i) = 1
\]

given \( \#Y^* = \#Z_i \) conditions equivalent

many private signals per public message (2) weaker: allows private signals to differ in how informative they are about the message the opposing player will send
Further Discussion

easier to satisfy with coarse message maps
vacuously satisfied if $m_1$ and $m_2$ are equal to the same constant
condition will have force when combined with assumption that messages “reveal enough” about the action profile that generated the underlying signals.
except in the trivial case of perfect information (2) rules out $z_1, z_2$ independent conditional on $a$
requires if one player receives a signal unlikely conditional on $a$, it is likely that the other player receive the corresponding unlikely signal
**Information Matrix**

consider \( \pi^m_{\alpha}(\cdot \mid Y^*) \) as row vector

construct a matrix \( \Pi^{m,i}_{\alpha} \) by stacking row vectors corresponding to \( (\tilde{a}_i, a_{-i}) \) as \( \tilde{a}_i \) ranges over \( A_i \)

stack two matrices corresponding to the two players to get a \((\#A_1 + \#A_2) \times \# Y^*\) matrix \( \Pi^m_{\alpha} \)

this matrix has two rows (both corresponding to \( \alpha \)) that are identical.

**Definition 2:** A game has **pure-strategy pairwise full rank** with respect to \( m \) if for every pure profile \( \alpha \) the rank of \( \Pi^m_{\alpha} \) is \( (\#A_1 + \#A_2) - 1 \).

never satisfied in games such as Green and Porter where players have the same sets of feasible actions, and the distribution of signals satisfies symmetry condition that \( \pi(\alpha, \alpha') = \pi(\alpha', \alpha) \)

is satisfied for set of probability measures \( \pi_{\alpha} \) of full Lebesgue measure.
Nash Threats Folk Theorem

$v^*$ be static Nash payoff vector normalized so $v^* = 0$

consider a sequence of games indexed by $n$

**Corollary:** Fix a message profile $m$, and suppose that $g^n \rightarrow g$, $\pi^n \rightarrow \pi$, that game $n$ has $(\varepsilon^n, \nu)$ public information with respect to $m$, that $\varepsilon^n \rightarrow 0$, that $\pi_a^{m}(\cdot | Y^*)$ has pure-strategy pairwise full rank with respect to $m$, and that each $g^n$ has a static equilibrium with payoffs converging to 0. Then there is a sequence $\gamma^n \rightarrow 0$ such that for any feasible interior vector of payoffs $v > 0$ there exists $\delta^* < 1$ and an $N$ such that for any $n > N$ and all $\delta \geq \delta^*$ there is a perfect public equilibrium in the game $n$ with payoffs $v^n$ satisfying $\|v^n - v\| < \gamma^n$. 
Idea of Proof

Find an auxiliary game where there is no disagreement

Prove a uniform version of the folk theorem in that public information game: using the arguments from Fudenberg, Levine, and Maskin [8] and McLean, Obara and Postlewaite [15]

Map back to the original game and punish players for disagreeing

Not so likely to disagree on equilibrium path
Use of Public Information

announcements are public information
why not use the regular folk theorem for that case?
FLM folk theorem limited to the convex hull of the set of profiles that satisfy enforceability plus pairwise identifiability
fix profile, including a strategy for sending messages
a player can randomize announcements independent of private information while preserving the marginal distribution of messages: “faking the marginal”
pairwise identifiability fails, because player one faking his marginal and player two faking hers are observationally equivalent
Information Aggregation

make same announcement for several different private signals.

two effects:

1. increases degree to which each player can forecast the other player’s message, reducing role of private information

2. reduces the informativeness of the messages, making it less likely that the assumption of pairwise full-rank is satisfied
Notions of Equilibrium

$\varepsilon$-sequential

every player following every of his private histories and public history has consistent beliefs such that conditional on his information he loses no more than $\varepsilon$ in average present value measured at that time by deviating

uniform equilibrium with respect to time averaging

1. time average converges on equilibrium path

2. for any $\rho > 0, \tau$ there exists $T > \tau$ such that any deviation loses at least $\rho$ in finite time $T$ average
Approximate Equilibrium and Time Averaging

Theorem: Suppose $T^n, \epsilon^n > 0, \sigma^n$ such that $\sigma^n$ is $T^n$ finite horizon $\epsilon^n$-approximate Nash equilibrium with payoff $v^n$, and that $\epsilon^n \to 0$, $v^n \to v$. Then:

A. There exist $\delta^n \to 1, \epsilon^n \to 0$, $\sigma^n$ such that $\sigma^n$ is $\epsilon^n$-sequential for $\delta^n$ and the equilibrium average present values converge to $v$

B. There exists a uniform equilibrium with payoff $v$
**Mutual threat point** a payoff vector \( v \) such that there exists a mutual punishment action: mixed action profile \( \alpha \) such that \( g_i(\alpha_i', \alpha_{-i}) \leq v \)

Consider enforceable mutually punishable set \( V^* \): intersection of closure of the convex hull of the payoff vectors that weakly pareto dominate a mutual punishment point and the closure of the convex hull of the enforceable payoffs

Difference with standard folk theorem: can exclude unenforceable actions and the minmax point may not be mutually punishable
Informational Connectedness

relevant only with more than two players

player $i$ is directly connected to player $j \neq i$ despite player $k \neq i, j$ if exists mixed profile $\alpha$ and mixed action $\hat{\alpha}_i$ such that

$$\pi_j(\cdot | \hat{\alpha}_i, \alpha_{k'}, \alpha_{-i-k}) \neq \rho_j(\alpha) \text{ for all } \alpha_{k'}.$$

$i$ is connected to $j$ if for every $k \neq i, j$ there is a sequence of players $i_1, \ldots, i_n$ with $i_1 = i, i_n = j$ and $i_p = k$ for any $p$ such that player $i_p$ is directly connected to player $i_{p+1}$ despite player $k$.

game is informationally connected if there are only two players, or if every player is connected to every other player.
Theorem 8.1: In an informationally connected game if \( v \in V^* \) then there exists a sequence of discount factors \( \delta_n \to 1 \), non-negative numbers \( \varepsilon_n \to 0 \) and strategy profiles \( \sigma_n \) such that \( \sigma_n \) is an \( \varepsilon_n \)-sequential equilibrium for \( \delta_n \) and equilibrium payoffs converge to \( v \).

- Use communication and punishment phases that are a small fraction of the total time
- Aggregate information over a long time before deciding what to do
- Need the epsilon so you if you’ve generated really good signals you don’t cheat as you approach the assessment phases
Belief Free Equilibrium and Friends

construct equilibria with the property that my best play does not depend on what I believe about your history

➤ gets around the coordination problem

➤ a possibly small subset of all equilibria

➤ but big enough that you can prove some folk theorems this way without communication