Long Run versus Short Run Player

*a fixed simultaneous move* stage game

Player 1 is long-run with discount factor $\delta$
actions $a^1 \in A^1$ a finite set
utility $u^1(a^1, a^2)$

Player 2 is short-run with discount factor 0
actions $a^2 \in A^2$ a finite set
utility $u^2(a^1, a^2)$

the “short-run” player may be viewed as a kind of “representative” of many “small” long-run players
- the “usual” case in macroeconomic/political economy models
- the “long run” player is the government
- the “short-run” player is a representative individual
Example 1: Peasant-Dictator
**Example 2: Backus-Driffil**

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>0,0</td>
<td>-2,-1</td>
</tr>
<tr>
<td>High</td>
<td>1,-1</td>
<td>-1,0</td>
</tr>
</tbody>
</table>

**Inflation Game:** LR=government, SR=consumers
consumer preferences are whether or not they guess right

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>0,0</td>
<td>0,-1</td>
</tr>
<tr>
<td>High</td>
<td>-1,-1</td>
<td>-1,0</td>
</tr>
</tbody>
</table>

with a hard-nosed government
Repeated Game

history $h_t = (a_1, a_2, \ldots, a_t)$
null history $h_0$
behavior strategies $\alpha_t^i = \sigma_t^i(h_{t-1})$
long run player preferences
average discounted utility

$$(1 - \delta) \sum_{t=1}^{T} \delta^{t-1} u^i(a_t)$$

note that average present value of 1 unit of utility per period is 1
Equilibrium

Nash equilibrium: usual definition – cannot gain by deviating
Subgame perfect equilibrium: usual definition, Nash after each history
Observation: the repeated static equilibrium of the stage game is a subgame perfect equilibrium of the finitely or infinitely repeated game
♦ strategies: play the static equilibrium strategy no matter what
“perfect equilibrium with public randomization”

may use a public randomization device at the beginning of each period to pick an equilibrium

key implication: set of equilibrium payoffs is convex
**Example: Peasant-Dictator**

normal form: unique Nash equilibrium \textbf{high, eat}

<table>
<thead>
<tr>
<th></th>
<th>eat</th>
<th>grow</th>
</tr>
</thead>
<tbody>
<tr>
<td>low</td>
<td>0*,1</td>
<td>1,2*</td>
</tr>
<tr>
<td>high</td>
<td>0*,1*</td>
<td>3*,0</td>
</tr>
</tbody>
</table>
payoff at static Nash equilibrium to LR player: 0

precommitment or Stackelberg equilibrium
precommit to low get 1
mixed precommitment to 50-50 get 2

minmax payoff to LR player: 0
utility to long-run player

mixed precommitment/Stackelberg = 2
best dynamic equilibrium = ?
pure precommitment/Stackelberg = 1
Set of dynamic equilibria
static Nash = 0
worst dynamic equilibrium = ?
minmax = 0
Repeated Peasant-Dictator

finitely repeated game
final period: high, eat, so same in every period
Do you believe this??

Infinitely repeated game

begin by low, grow
if low, grow has been played in every previous period then play low, grow
otherwise play high, eat (reversion to static Nash)
claim: this is subgame perfect
clearly a Nash equilibrium following a history with high or eat
SR play is clearly optimal

for LR player
may high and get
or low and get 1

so condition for subgame perfection

\[(1 - \delta)3 \leq 1\]
\[\delta \geq 2/3\]
equilibrium utility for LR

\[ \delta \frac{2}{3} \]

\[ \frac{1}{1} \]
General Deterministic Case (Fudenberg, Kreps and Maskin)

\[ \max u^1(a) \]

- mixed precommitment/Stackelberg
- \( v^1 \) best dynamic equilibrium
- pure precommitment/Stackelberg

Set of dynamic equilibria

- static Nash
- \( v^1 \) worst dynamic equilibrium
- minmax
- \( \min u^1(a) \)
Characterization of Equilibrium Payoff

\( \alpha = (\alpha^1, \alpha^2) \) where \( \alpha^2 \) is a b.r. to \( \alpha^1 \)

\( \alpha \) represent play in the first period of the equilibrium

\( w^1(a^1) \) represents the equilibrium payoff beginning in the next period

\[
\begin{align*}
v^1 &\geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1) \\
v^1 &\leq w^1(a^1) \leq \bar{v}^1
\end{align*}
\]

strategy: impose stronger constraint using \( n \) static Nash payoff

for best equilibrium \( n \leq w^1(a^1) \leq \bar{v}^1 \)

for worst equilibrium \( \underline{v}^1 \leq w^1(a^1) \leq n \)

avoids problem of best depending on worst
remark: if we have static Nash = minmax then no computation is needed for the worst, and the best calculation is exact.
max problem
fix \( \alpha = (\alpha^1, \alpha^2) \) where \( \alpha^2 \) is a b.r. to \( \alpha^1 \)

\[
\bar{v}^1 \geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1)
\]

\[
\bar{v}^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1), \alpha^1(a^1) > 0
\]

\[
n^1 \leq w^1(a^1) \leq \bar{v}^1
\]

how big can \( w^1(a^1) \) be in \( = \) case?

Biggest when \( u^1(a^1, \alpha^1) \) is smallest, in which case

\[
w^1(a^1) = \bar{v}^1
\]

\[
\bar{v}^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta \bar{v}^1
\]

conclusion for fixed \( \alpha \)
\[
\min_{a^1 | \alpha(a^1) > 0} u^1(a^1, \alpha^2)
\]

i.e. worst in support

\[
\bar{v}^1 = \max_{\alpha^2 \in BR^2(a^1)} \min_{a^1 | \alpha(a^1) > 0} u^1(a^1, \alpha^2)
\]

observe:

mixed precommitment \(\geq \bar{v}^1 \geq\) pure precommitment
Peasant-Dictator Example

<table>
<thead>
<tr>
<th></th>
<th>eat</th>
<th>grow</th>
</tr>
</thead>
<tbody>
<tr>
<td>low</td>
<td>0*,1</td>
<td>1,2*</td>
</tr>
<tr>
<td>high</td>
<td>0*,1*</td>
<td>3*,0</td>
</tr>
</tbody>
</table>

\[ p(\text{low}) \] \[ \begin{array}{|c|c|c|} \hline p(\text{low}) & \text{BR} & \text{worst in support} \\ \hline 1 & \text{grow} & 1 \\ \frac{1}{2} < p < 1 & \text{grow} & 1 \\ p = \frac{1}{2} & \text{any mixture} & \leq 1 \ (\text{low}) \\ 0 < p < \frac{1}{2} & \text{eat} & 0 \\ p = 0 & \text{eat} & 0 \\ \hline \end{array} \]
check: 

\[ w^1(a^1) = \frac{\bar{v}^1 - (1 - \delta)u^1(a^1, \alpha^2)}{\delta} \geq n^1 \]

as \( \delta \to 1 \) then \( w^1(a^1) \to \bar{v}^1 \geq n^1 \)
The min problem

Fix \( \alpha = (\alpha^1, \alpha^2) \) where \( \alpha^2 \) is a b.r. to \( \alpha^1 \)

\[
\begin{align*}
v^1 & \geq (1 - \delta)u^1(a^1, \alpha^2) + \delta w^1(a^1) \\
v^1 & \leq w^1(a^1) \leq n^1
\end{align*}
\]

Biggest \( u^1(a^1, \alpha^1) \) must have smallest \( w^1(a^1) = v^1 \)

\[
v^1 = (1 - \delta)u^1(a^1, \alpha^2) + \delta v^1
\]

Conclusion

\[
v^1 = \max u^1(a^1, \alpha^2)
\]

Or

\[
v^1 = \min_{\alpha^2 \in BR^2(\alpha^1)} \max u^1(a^1, \alpha^2)
\]

That is, constrained minmax
### Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0,-3</td>
<td>1,2</td>
<td>0,3</td>
</tr>
<tr>
<td>D</td>
<td>0,3*</td>
<td>2,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

static Nash gives 0  
minmax gives 0  
worst payoff in fact is 0  
pure precommitment also 0
mixed precommitment

$p$ is probability of up

to get more than 0 must get SR to play M

$-3p + (1-p)3 \leq 2$ and $3p \leq 2$

first one

$-3p + (1-p)3 \leq 2$

$-3p - 3p \leq -1$

$p \geq 1/6$

second one
\[ 3p \leq 2 \]
\[ p \leq \frac{2}{3} \]

want to play D so take \( p = \frac{1}{6} \)

get \( \frac{1}{6} + \frac{10}{6} = \frac{11}{6} \)
utility to long-run player

\[ \max u^1(a) = 2 \]

mixed precommitment/Stackelberg = 11/16

\[ \bar{v}^1 \] best dynamic equilibrium = 1

pure precommitment/Stackelberg = 0

Set of dynamic equilibria

static Nash = 0

\[ v^1 \] worst dynamic equilibrium = 0

minmax = 0

\[ \min u^1(a) = 0 \]
calculation of best dynamic equilibrium payoff

$p$ is probability of up

<table>
<thead>
<tr>
<th>$p$</th>
<th>$BR^2$</th>
<th>worst in support</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;1/6$</td>
<td>L</td>
<td>0</td>
</tr>
<tr>
<td>$1/6&lt;p&lt;5/6$</td>
<td>M</td>
<td>1</td>
</tr>
<tr>
<td>$p&gt;5/6$</td>
<td>R</td>
<td>0</td>
</tr>
</tbody>
</table>

so best dynamic payoff is 1