Questions on Learning and Evolution in Games

Instructions: Answers should be typed and formatted as a pdf file. Your explanation of your answers and your clarity are important. You should be brief but accurate and clear.

1. Risk Dominance and Pareto Efficiency Suppose that $x \leq 21$. The symmetric game

	B	G
B	20,20	19,x
G	x,19	21,21

   has a Nash equilibrium that Pareto dominates all other outcomes of the game, plus another pure Nash equilibrium. For what values of x is the Pareto dominant equilibrium also risk dominant?

2. In a two-player game, a marginal best response distribution is a probability distribution over outcomes (possibly correlated) such that each player has expected utility at least as much as that from the best response to the marginal distribution over his opponents actions. Show that in a 2x2 game a marginal best best response distribution is a correlated equilibrium.

3. (a) Starting with an initial condition in which each player has observed one heads, calculate the first ten periods of fictitious play in matching pennies. Assume that in case of a tie, a player plays the same way he did the previous period. What is the frequency each player played heads?
   (b) Consider a two state Markov process in which there is a 3/4 chance of remaining in state 1 but only a 1/2 chance of remaining in state 2. What is the unique stationary distribution? What does this mean about the long run frequency with which state 1 is observed?

4. Consider the one population continuous time replicator dynamic for the game with payoffs

	R	P	S
R	0,0	0,2	2,0
P	2,0	0,0	0,2
S	0,2	2,0	0,0

   (a) Find the unique Nash equilibrium of this game. (b) Find the eigenvalues for the replicator at that Nash equilibrium. (c) What does this tell you about the dynamical system?

5. Consider the "exponential" best response dynamic for a heterogeneous population model where the rate of change of the proportion of population of type i is given by

$$\dot{\theta}^i(s^i) = \frac{\exp \kappa^i u^i(s^i, \theta^{-i})}{\sum_{\tilde{s}^i} \exp \kappa^i u^i(\tilde{s}^i, \theta^{-i})} - \theta^i(s^i)$$

   (a) What happens as $\kappa^i \to \infty$? (b) Show that for any $\kappa^i < \infty$ this dynamical system is volume contracting (hint: linearize the system; it suffices to show that the trace of the resulting matrix is negative). (c) What does the fact that the system is volume contracting tell you about the dynamics of the system?

6. You will analyze using evolutionary tools the symmetric game below.

	A	B	C	D
A	3,3	0,0	0,0	0,-1
B	0,0	2,2	0,0	0,-1
C	0,0	0,0	1,1	5,-1
D	-1,0	-1,0	-1,5	-1,-1

   (a) What are all the pure and mixed strategy equilibria of this game? (Hint: there should be three pure equilibria and four equilibria involving non-trivial mixtures.)
   The dynamic process is evolutionary and indexed by ϵ. There is a finite population of players. Each period two players are chosen at random to play. The state variable is the number of players who most recently used each strategy. It evolves following the dynamic
   (i) With probability ϵ a player "mutates" meaning she chooses randomly with equal weights among all of his strategies.

(ii) With probability $1 - \epsilon$ a player chooses randomly with equal weights among all best responses to the current population distribution of strategies.

(b) The current population distribution of strategies includes the strategy used by the player herself. Does this make sense? When does it make a difference? What is the advantage of this approach?

Consider first the case where $\epsilon = 0$, that is with probability 1 a player chooses randomly with equal weight among all best responses to the current population distribution of strategies.

(c) Can there be a best-response cycle (Hint: consider increasing A,B or C by one – can you make an argument involving monotonicity?)

Now suppose that the population is finite with $n = 11^2 = 121$

(d) Show that the only configurations of the state corresponding to Nash equilibria represent the three pure strategy equilibria.

Now consider the case with $\epsilon > 0$.

(e) Create a table with the least resistance between each ordered pair of pure strategy equilibria. (Remark: remember you have to consider all possible paths from one to the other and find the path with the least resistance.)

(f) Using the table you created in (e) analyze all the possible trees and determine which are the least resistance trees. What can you conclude about stochastically stable states?

(g) Redo part (f) using instead the method of circuits.

(h) The strategy D is strictly dominated. Suppose that when a mutation occurs instead of choosing randomly among all strategies D is chosen with probability and the remaining weight is distributed equally among the other three strategies. What difference would that make?