Strategic Form Games
Finite Games

an $N$ player game $i = 1 \ldots N$

$\Delta(S)$ are probability measure on $S$

finite strategy spaces $S_i$

$\sigma_i \in \Sigma_i \equiv P(S_i)$ are mixed strategies

$s \in S \equiv \times_{i=1}^{N} S_i$ are the strategy profiles, $\sigma \in \Sigma \equiv \times_{i=1}^{N} \Sigma_i$

other useful notation $s_{-i} \in S_{-i} \equiv \times_{j \neq i} S_j$

$\sigma_{-i} \in \Sigma_{-i} \equiv \times_{j \neq i} \Sigma_j$

$u_i(s)$ payoff or utility

$u_i(\sigma) \equiv \sum_{s \in S} u_i(s) \prod_{j=1}^{N} \sigma_j(s_j)$ is expected utility
Dominance and Rationalizability

\( \sigma_i \) weakly (strongly) dominates \( \sigma'_i \) if

\[ u_i(\sigma_i, s_{-i}) \geq (>)u_i(\sigma'_i, s_{-i}) \]

with at least one strict

Prisoner’s Dilemma Game

fire in the theater version

<table>
<thead>
<tr>
<th></th>
<th>orderly</th>
<th>run for exit</th>
</tr>
</thead>
<tbody>
<tr>
<td>orderly</td>
<td>2,2</td>
<td>0,3</td>
</tr>
<tr>
<td>run for exit</td>
<td>3,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

a unique dominant strategy equilibrium (run,run); pareto dominated by (orderly, orderly)
Public Goods Experiment

Players randomly matched in pairs

May donate or keep a token

The token has a fixed commonly known public value of 15

It has a randomly drawn private value uniform on 10-20

V=private gain/public gain

So if the private value is 20 and you donate you lose 5, the other player gets 15; V= -1/3

If the private value is 10 and you donate you get 5 the other player gets 15; V=+1/3

Data from Levine/Palfrey, experiments conducted with caltech undergraduates

Based on Palfrey and Prisbey
<table>
<thead>
<tr>
<th>V</th>
<th>donating a token</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>100%</td>
</tr>
<tr>
<td>0.2</td>
<td>92%</td>
</tr>
<tr>
<td>0.1</td>
<td>100%</td>
</tr>
<tr>
<td>0</td>
<td>83%</td>
</tr>
<tr>
<td>-0.1</td>
<td>55%</td>
</tr>
<tr>
<td>-0.2</td>
<td>13%</td>
</tr>
<tr>
<td>-0.3</td>
<td>20%</td>
</tr>
</tbody>
</table>
Second Price Auction

A single item is to be auctioned. Value to the seller is zero.

\[ i = 1, \ldots, N \] buyers

Value \( v_i > 0 \) to buyer \( i \).

Each buyer submits a bid \( b_i \).

The item is sold to the highest bidder at the second highest bid.

Bidding your value weakly dominates.

Connection to the Becker-DeGroot-Marschak elicitation procedure and WTAccept, WTPay.
## Iterated Dominance

example of iterated weak dominance

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R-l</th>
<th>R-r</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-u</td>
<td>-1,-1</td>
<td>2,0</td>
<td>1,1</td>
</tr>
<tr>
<td>U-d</td>
<td>-1,-1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>1,1</td>
<td>1,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Eliminate U-d
Eliminate L
Eliminate D (or) Eliminate R-l
Eliminate R-I

Notice that there can be more than one answer for iterated weak dominance

Not for iterated strong dominance
Cournot Oligopoly

\[ \pi_i = [17 - (x_i + x_{-i})]x_i - x_i; \text{ reaction function } x_i = 8 - \frac{x_{-i}}{2} \]

\[ x_1 = 8 - \frac{x_2}{2} \]

\[ x_2 = 8 - \frac{x_1}{2} \]
Nash Equilibrium

“no further learning is possible”

\( \sigma \) is a *Nash equilibrium* profile if for each \( i \in 1, \ldots, N \)
\[
    u_i(\sigma) = \max_{\sigma'_i} u_i(\sigma'_i, \sigma_{-i})
\]
Theorem: a Nash equilibrium exists in a finite game

This theorem fails in pure strategies: consider matching pennies Holmes and Moriarity
this is more or less why Kakutani’s fixed point theorem was invented:

An upper hemi-continuous (UHC) convex valued correspondence $B$ from a convex subset $\Sigma \subseteq \mathbb{R}^n$ to itself has a fixed point $\sigma \in B(\sigma)$

A correspondence $B : \Sigma \rightrightarrows \Sigma$ is UHC means if $\sigma^n \to \sigma$ such that $b^n \in B(\sigma^n) \to b$ then $b \in B(\sigma)$. 
Proof: Let $B_i(\sigma)$ be the set of best responses of $i$ to $\sigma_{-i}$

convex valued: convex combinations of a best response is a best response. Specifically, since you must be indifferent between all pure strategies played with positive probability, the best response set is the set of all convex combinations of the pure strategies that are best responses.

UHC: $b^n_i \in B_i(\sigma^n_{-i}) \rightarrow b_i$ means that $u^i(b^n_i, \sigma^n_{-i}) \geq u^i(\sigma_i, \sigma^n_{-i})$. Suppose the converse that $b \not\in B(\sigma)$. This means for some $\sigma_{i}$ that $u^i(\sigma_{i}, \sigma_{-i}) > u^i(b_i, \sigma_{-i})$. Since $\sigma^n_{-i} \rightarrow \sigma_{-i}$ for $n$ sufficiently large, since $u^i$ is continuous (multi-linear in fact) in $\sigma_{-i}$, $u^i(\sigma^n_{i}, \sigma^n_{-i}) > u^i(b^n_i, \sigma^n_{-i})$. Since $b^n_i \rightarrow b_i$, since $u^i$ is continuous (linear in fact) in $\sigma_i$, also for $n$ sufficiently large $u^i(\sigma^n_{i}, \sigma^n_{-i}) > u^i(b^n_i, \sigma^n_{-i})$. This contradicts $b^n_i \in B_i(\sigma^n_{-i})$.

“a sequence of best-responses converges to a best-response”
Mixed Strategies: The Kitty Genovese Problem

Description of the problem

Model:

\( n \) people all identical
benefit if someone calls the police is \( x \)
cost of calling the police is 1
Assumption: \( x > 1 \)

Look for symmetric mixed strategy equilibrium where \( p \) is probability of each person calling the police
\( p \) is the symmetric equilibrium probability for each player to call the police

each player \( i \) must be indifferent between calling the police or not
if \( i \) calls the police, gets \( x - 1 \) for sure.
If \( i \) doesn’t, gets 0 with probability \((1 - p)^{n-1}\), gets \( x \) with probability \(1 - (1 - p)^{n-1}\)
so indifference when \( x - 1 = x \left(1 - (1 - p)^{n-1}\right)\)
solve for \( p = 1 - \left(1 / x\right)^{1/(n-1)}\)
probability police is called

\[
1 - (1 - p)^n = 1 - \left(\frac{1}{x}\right)^{n/(n-1)}
\]
The probability police are called.
Do Better People Make a Better Society?

“If we were all better people the world would be a better place”

may seem to you to be self-evidently true

involves the fallacy of composition: just because a statement applies to each individual person it need not apply to the group
### Pride Game

<table>
<thead>
<tr>
<th></th>
<th>Proud</th>
<th>Not confess</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proud</td>
<td>40,40</td>
<td>54,36</td>
<td>12,0</td>
</tr>
<tr>
<td>Not confess</td>
<td>36,54</td>
<td>50,50</td>
<td>-40,100</td>
</tr>
<tr>
<td>Confess</td>
<td>0,12</td>
<td>100,-40</td>
<td>10,10</td>
</tr>
</tbody>
</table>

Yellow cells are a Prisoner’s Dilemma game

Proud individual will not confess except in retaliation against a rat-like opponent who confesses

Not confess dominated by Proud

Iterating: Confess dominate by Proud

So unique solution of iterated strict dominance in blue
“Better People”

more caring about each other
more altruistic
place a value on the utility received in “selfish” game by both players
not complete altruism “better people” not “perfect people”
twice as much weight on own utility as on opponent
(30,60) in selfish game gives player 1 \((2/3)*30+(1/3)*60=40\) in altruistic game
### Altruistic Pride Game

<table>
<thead>
<tr>
<th></th>
<th>Proud</th>
<th>Not confess</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proud</td>
<td>40,40</td>
<td>48,42*</td>
<td>8,4</td>
</tr>
<tr>
<td>Not confess</td>
<td>42*,38</td>
<td>50,50</td>
<td>6.7,53.3*</td>
</tr>
<tr>
<td>Confess</td>
<td>4,8</td>
<td>53.3*,6.7</td>
<td>10*,10*</td>
</tr>
</tbody>
</table>
Mixed Equilibria

Compute using the open source software program Gambit written by Richard McKelvey, Andrew McLennan and Theodore Turocy

- randomizing between proud and confess: so is worse than proud-proud

- strictly mixed: gives each player 23.1, better than both confessing for certain, still less good than unique equilibrium of the Pride Game.

- conclusion: in this example better people make the world an unambiguously worse place
Practical Aspects

The game is contrived to make a point

- more altruistic would choose to forgive and forget more criminal behavior
- more altruistic criminals would choose commit fewer crimes
- crime not punished so severely, they would be inclined to commit more crimes
- effect on crime ambiguous
- if more crimes are committed, the world can be a worse place

based on Hwang and Bowles [2009]
Coordination Games

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>R</td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

three equilibria (L,L) (R,R) (.5L,.5R)
the Swedish experience

**too many equilibria?? introspection possible?**

the rush hour traffic game – introspection clearly impossible, yet we seem to observe Nash equilibrium

equilibrium through learning?
Coordination Experiments
Van Huyck, Battalio and Beil [1990]

Actions \( A = \{1, 2, \ldots, \overline{e}\} \)

Utility \( u(a_i, a_{-i}) = b_0 \min(a_j) - ba_i \) where \( b_0 > b > 0 \)

Everyone doing \( a' \) the same thing is always a Nash equilibrium

\( a' = \overline{e} \) is efficient

the bigger is \( a' \) the more efficient, but the “riskier”

a model of “riskier” some probability of one player playing \( a' = 1 \)

story of the stag-hunt game
\[ \bar{e} = 7, \ 14-16 \text{ players} \]
treatments: \[ A \quad b_0 = 2b \]
\[ B \quad b = 0 \]

In final period treatment A:
77 subjects playing \( a_i = 1 \)
30 subjects playing something else
minimum was always 1

In final period treatment B:
87 subjects playing \( a_i = 7 \)
0 playing something else
with two players \( a_i = 7 \) was more common
**1/2 Dominance**

do the same as half of population

Coordination Game

<table>
<thead>
<tr>
<th></th>
<th>L ( p_2 = 11/13 )</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2</td>
<td>-10,0</td>
</tr>
<tr>
<td>( p_1 = 11/13 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>0,-10</td>
<td>1,1[1/2 dom]</td>
</tr>
</tbody>
</table>

(In 2x2 games same as risk dominance)

indifference between U,D

\[ 2p_2 - 10(1 - p_2) = (1 - p_2) \]

\[ 13p_2 = 11, p_2 = 11/13 \]
Trembling Hand Perfection

$\sigma$ is trembling hand perfect if there is a sequence $\sigma^n >> 0, \sigma^n \rightarrow \sigma$ such that

if $\sigma^i(s^i) > 0$ then $s^i$ is a best response to $\sigma^n$

Note: thp is necessarily a Nash equilibrium

Examples:
strict Nash equilibrium is always thp
completely mixed Nash equilibrium is always thp
### Correlated Equilibrium

#### Chicken

<table>
<thead>
<tr>
<th></th>
<th>6,6</th>
<th>2,7</th>
</tr>
</thead>
<tbody>
<tr>
<td>7,2</td>
<td></td>
<td>0,0</td>
</tr>
</tbody>
</table>

Three Nash equilibria (2,7), (7,2) and mixed equilibrium with probabilities (2/3, 1/3) and payoffs (4 2/3, 4 2/3)
What is public randomization?

<table>
<thead>
<tr>
<th>6,6</th>
<th>2,7</th>
</tr>
</thead>
<tbody>
<tr>
<td>7,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

correlated strategy

<table>
<thead>
<tr>
<th>1/3</th>
<th>1/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>0</td>
</tr>
</tbody>
</table>

is a correlated equilibrium giving utility (5,5)
Approximate Equilibria and Near Equilibria

- exact: \[ u_i(s_i | \sigma_{-i}) \geq u_i(s'_i | \sigma_{-i}) \]

approximate: \[ u_i(s_i | \sigma_{-i}) + \varepsilon \geq u_i(s'_i | \sigma_{-i}) \]

- Approximate equilibrium can be very different from exact equilibrium

Radner’s work on finite repeated PD
gang of four on reputation

upper and lower hemi-continuity
A small portion of the population playing "non-optimally" may significantly change the incentives for other players causing a large shift in equilibrium behavior.
Quantal Response Equilibrium
(McKelvey and Palfrey)

Individual Play in Voting
propensity to play a strategy

\[ p_i(s_i) = \exp(\lambda_i u_i(s_i, \sigma_{-i})) \]

\[ \sigma_i(s_i) = \frac{p_i(s_i)}{\sum_{s_i'} p_i(s_i')} \]

as \( \lambda_i \to \infty \) approaches best response

as \( \lambda_i \to 0 \) approaches uniform distribution
Smoothed Best Response Correspondence Example

\[
\begin{array}{c|c}
L (\sigma_2(L) = q) & R \\
\hline
U (\sigma_1(U) = p) & 1,1 & 0,0 \\
D & 0,0 & 1,1 \\
\end{array}
\]
Goeree and Holt: Matching Pennies

Symmetric

<table>
<thead>
<tr>
<th></th>
<th>50% (48%)</th>
<th>50% (52%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50% (48%)</td>
<td>80,40</td>
<td>40,80</td>
</tr>
<tr>
<td>50% (52%)</td>
<td>40,80</td>
<td>80,40</td>
</tr>
<tr>
<td></td>
<td>12.5% (16%)</td>
<td>87.5% (84%)</td>
</tr>
<tr>
<td>----------------</td>
<td>-------------</td>
<td>-------------</td>
</tr>
<tr>
<td>50% (96%)</td>
<td>320,40</td>
<td>40,80</td>
</tr>
<tr>
<td>50% (4%)</td>
<td>40,80</td>
<td>80,40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(80%)</th>
<th>(20%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50% (8%)</td>
<td>44,40</td>
<td>40,80</td>
</tr>
<tr>
<td>50% (92%)</td>
<td>40,80</td>
<td>80,40</td>
</tr>
</tbody>
</table>