Bayesian Games and Mechanism Design

**Definition of Bayes Equilibrium**

Harsanyi [1967]

- What happens when players do not know one another’s payoffs?
- Games of “incomplete information” versus games of “imperfect information”
- Harsanyi’s notion of “types” encapsulating “private information”
- Nature moves first and assigns each player a type; player’s know their own types but not their opponents’ types
- Players do have a common prior belief about opponents’ types
Bayesian Games

There are a finite number of types $\theta_i \in \Theta_i$

There is a common prior $p(\theta)$ shared by all players

$p(\theta_{-i} | \theta_i)$ is the conditional probability a player places on opponents’ types given his own type

The stage game has finite action spaces $a_i \in A_i$ and has utility functions $u^i(a, \theta)$
Bayesian Equilibrium

A *Bayesian Equilibrium* is a Nash equilibrium of the game in which the strategies are maps from types $s_i : \Theta_i \rightarrow A_i$ to stage game actions $A_i$.

This is equivalent to each player having a strategy as a function of his type $s_i(\theta_i)$ that maximizes conditional on his own type $\theta_i$ (for each type that has positive probability)

$$\max_{s_i} \sum_{\theta_{-i}} u_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) p(\theta_{-i} | \theta_i)$$
Cournot Model with Types

- A duopoly with demand given by $p = 17 - x$
- A firm’s type is its cost, known only to that firm: each firm has a 50-50 chance of cost constant marginal cost 1 or 3.

profits of a representative firm

$$\pi_i(c_i, x) = [17 - c_i - (x_i + x_{-i})] x_i$$

Let us look for the symmetric pure strategy equilibrium
Finding the Bayes-Nash Equilibrium

$x^1, x^3$ will be the output chosen in response to cost

$$\pi_i(x_i, c_i) = .5 \left[ 17 - c_i - (x_i + x^1) \right] x_i$$

$$+ .5 \left[ 17 - c_i - (x_i + x^3) \right] x_i$$

maximize with respect to $x_i$ and solve to find

$x^1 = 11/2$, $x^3 = 9/2$
industry output
probability \( \frac{1}{4} \) 11
probability \( \frac{1}{2} \) 10
probability \( \frac{1}{4} \) 9

Suppose by contrast costs are known
If both costs are 1 then competitive output is 16 and Cournot output is 2/3rds this amount 10 2/3
If both costs are 3 then competitive output is 14 and Cournot output is 9 1/3
If one cost is 1 and one cost is 3 Cournot output is 10
With known costs, mean industry output is the same as with private costs, but there is less variation in output
Sequentiality and Signaling

Cho-Kreps [1987]
**Sequentiality**
Kreps-Wilson [1982]

Subforms

Beliefs: *assessment* $a_i$ for player $i$ probability distribution over nodes at each of his information sets; *belief* for player $i$ is a pair $b_i \equiv (a_i, \pi_{-i}^i)$, consisting of $i$’s assessment over nodes $a_i$, and $i$’s expectations of opponents’ strategies $\pi_{-i}^i = (\pi_{j}^i)_{j \neq i}$

Beliefs come from strictly positive perturbations of strategies

belief $b_i \equiv (a_i, \pi_{-i}^i)$ is *consistent* (Kreps and Wilson [17]) if $a_i = \lim_{n \to \infty} a_i^n$ where $a_i^n$ obtained using Bayes rule on a sequence of strictly positive strategy profiles of the opponents, $\pi_{-i}^{i,m} \to \pi_{-i}$
given beliefs we have a well-defined decision problem at each information set; can define optimality at each information set

A sequential equilibrium is a behavior strategy profile $\pi$ and an assessment $a_i$ for each player such that $(a_i, \pi^i_{-i})$ is consistent and each player optimizes at each information set.
Chain Store Paradox
Kreps-Wilson [1982], Milgrom-Roberts [1982]

finitely repeated model with long-run versus short-run
Reputational Model

two types of long-run player $\omega \in \Omega$

“rational type” and “committed type”

“committed type” will fight no matter what

types are privately known to long-run player, not known to short run player

Kreps-Wilson; Milgrom-Roberts

Solve for the sequential equilibrium; show that at the time-horizon grows long we get no entry until near the end of the game

“triumph of sequentiality”
The Holdup Problem

♦ Chari-Jones, the pollution problem
♦ problem of too many small monopolies

\[ \rho \] is the profit generated by an invention with a monopoly with a patent, drawn from a uniform distribution on \([0, 1]\), private to the inventor

\[ \phi^F \] is the fraction of this profit that can be earned without a patent

To create the invention requires as input \( N \) other existing inventions

It costs \( \varepsilon / N \) to make copies of these other inventions, where \( \varepsilon < 1/2 \) and \( \varepsilon / \phi^F < 1 \)
Case 1: Competition

if $\phi F \rho \geq \varepsilon$ the new invention is created, probability is $1 - \varepsilon / \phi F$.

Case 2: Patent

Each owner of the existing inventions must decide a price $p_i$ at which to license their invention; $\phi N$ current inventions are still under patent

Subgame Perfection/Sequentiality implies that the new invention is created when $(\phi + \phi F) \rho \geq \sum_i p_i + \varepsilon$

Profit of preexisting owners $(1 - \frac{(\phi N - 1)p + p_i - \varepsilon}{\phi + \phi F})p_i$

FOC $1 - \frac{(\phi N - 1)p + 2p_i + \varepsilon}{\phi + \phi F} = 0$

symmetric equilibrium $p = (\phi + \phi F - \varepsilon)/(\phi N + 1)$; $\sum_i p_i = \phi N p$

corresponding probability of invention $(\phi + \phi F - \varepsilon)/[(\phi + \phi F)(\phi N + 1)]$
Micro Mechanism Design

An “auction” problem

• Single seller has a single item
• Seller does not value item
• Two buyers with independent valuations

\[0 \leq v^l < v^h\] low and high valuations

\[\pi^l + \pi^h = 1\] probabilities of low and high valuations
what is the best way to sell the object

- Auction
- Fixed price
- Other
The Revelation Principle

Design a game for the buyers to play
• Auction game
• Poker game
• Etc.

Design the game so that there is a Nash equilibrium that yields highest possible revenue to the seller

The revelation principle says that it is enough to consider a special game
• strategies are “announcements” of types
• the game has a “truthful revelation” equilibrium
In the Auction Environment

Fudenberg and Tirole section 7.1.2

$q^l, q^h$ probability of getting item when low and high

$p^h, p^l$ expected payment when low and high
**individual rationality constraint**

(IR) \[ q_i^i v_i^i - p_i^i \geq 0 \]

- if you announce truthfully, you get at least the utility from not playing the game

**incentive compatibility constraint**

(IC) \[ q_i^i v_i^i - p_i^i \geq q_i^{-i} v_i^{-i} - p_i^{-i} \]

- you gain no benefit from lying about your type

the incentive compatibility constraint is the key to equilibrium
\textit{Other constraints}

$q^l, q^h$ probability of getting item when low and high
they can’t be anything at all:

probability constraints

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(win against other type, 50% chance of winning against self)

(2) $\pi^l q^l + \pi^h q^h \leq 1 / 2$
(probability of getting the good before knowing type less than 50%)
**Seller Problem**

Maximize seller utility $U = \pi^l p^l + \pi^h p^h$

Subject to IC and IR

To solve the problem we make a guess:

IR binds for low value
\[ q^l v^l - p^l = 0 \]

IC binds for high value
\[ q^h v^h - p^h = q^l v^h - p^l \]
The solution

\[ p^l = q^l v^l \text{ from low IR} \]

substitute into high IC

\[ p^h = (q^h - q^l)v^h + q^l v^l \]

plug into utility of seller

\[
U = \pi^l q^l v^l + \pi^h \left( (q^h - q^l)v^h + q^l v^l \right) \\
U = q^l (\pi^l v^l - \pi^h v^h + \pi^h v^l) + \pi^h q^h v^h \\
\pi^l + \pi^h = 1 \text{ so} \\
U = q^l \left( v^l - \pi^h v^h \right) + \pi^h q^h v^h
\]
Case 1: $v^l > \pi^h_v^h$

\[ U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h \]

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \leq 1 / 2$

Make $q^l, q^h$ large as possible so

\[ \pi^l q^l + \pi^h q^h = 1 / 2 \]

\[ U = \frac{1 / 2 - \pi^h q^h}{\pi^l} (v^l - \pi^h v^h) + \pi^h q^h v^h \]

\[ U = \frac{1}{2\pi^l} (v^l - \pi^h v^h) + q^h \frac{\pi^h}{\pi^l} (v^h - v^l) \]
so $q^h$ should be as large as possible

$$q^h = \pi^l + \pi^h / 2$$

plug back into (2) to find

$$q^l = \pi^l / 2$$

expected payments

$$p^l = q^l v^l, \quad p^h = (q^h - q^l)v^h + q^l v^l$$

$$p^l = v^l \pi^l / 2$$

$$p^h = v^h / 2 + \pi^l v^l / 2$$
Implementation of Case 1

modified auction: each player announces their value
the highest announced value wins
if there is a tie, flip a coin
if the low value wins, he pays his value
if the high value wins he pays

\[
\frac{p^h}{q^h} = \frac{v^h/2 + \pi^l v^l/2}{\pi^l + \pi^h/2}
\]

under these rules
probability that high type wins is \( q^h = \pi^l + \pi^h / 2 \)
probability that low type wins is \( q^l = \pi^l / 2 \)
just as in the optimal mechanism

this means the expected payments are the same too
Case 2: $v^l < \pi^h v^h$

\[ U = q^l (v^l - \pi^h v^h) + \pi^h q^h v^h \]

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \leq 1 / 2$

Make $q^h$ large as possible, $q^l$ as small as possible

$q^h = \pi^l + \pi^h / 2$

$q^l = 0$
expected payments

\[ p^l = q^l v^l, \quad p^h = (q^h - q^l)v^h + q^l v^l \]

\[ p^l = 0 \]

\[ p^h = (\pi^l + \pi^h / 2)v^h \]
Implementation of Case 2

set a fixed price equal to the highest valuation

\[ v^h = \frac{p^h}{q^h} = \frac{(\pi^l + \pi^h / 2)v^h}{\pi^l + \pi^h / 2} \]