The Concept of Income in a General Equilibrium
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J.A. Sefton and M.R. Weale[1]
National Institute of Economic and Social Research,
London SW1P 3HE

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Abstract

This paper derives a concept of aggregate income, which far from being a purely macro-economic concept or one relying on a representative consumer, can be linked to aggregate welfare in a general equilibrium economy consisting of finitely many hetergeneous infinitely-lived agents. This measure emerges naturally from an examination of the production side of the general equilibrium, but is also consistent with a definition of income as the level of consumption at which the presented discounted sum of current and future consumption is unchanging- a definition consistent with Hicks' idea that income is the level of consumption possible which leave the consumer as well off at the end of the week as he had been at the start.

Our measure of income, which we call Real Income resolves a number of previously unsatisfactorily unanswered questions. First we show how the concept of income relates to current and future consumption explaining the difference between this and measures of income based on sustainability; secondly we show that there is a simple additive link between the incomes of households and a measure of national income or social income; finally we show how the concept of national income relates to national product in both closed and open economies. Our framework also allows us to link capital gains to expected future changes in factor prices and show, at least in the economy as we model, it that capital gains should not be added on to income as conventionally measured. Our income measure meets two important invariance conditions which conventional national accounting measures do not fulfil. First Real Income is unaffected by whether a country with an excess of one factor of production addresses this by exporting the factor or by exporting goods produced using the factor intensively. Secondly the Real Income of a resource owner is the same whether the owner extracts the resource gradually or sells the extraction rights and draws income from investing the proceeds.

1 Introduction

Hicks (1939) defined income as "the maximum amount a man can spend and still be as well off at the end of the week as at the beginning". Since then there has been a succession of attempts to apply this definition. The problem is, as Hicks acknowledged, that the correct interpretation of "as well off" is by no means clear. Weitzman (1976) and Asheim (1994) suggest that income is equal to the level of consumption which could be sustained indefinitely out of the capitalised value of current income and equate this to being as well off at the end of the week as at the beginning. Eisner (1988*a*) argues specifically that in a "total incomes system" effects arising from asset price changes and conventionally regarded as capital gains should be included with income. Here, building on earlier work by Sefton & Weale (1996), Weitzman & Asheim (2001) and Pemberton & Ulph (2001) we provide a comprehensive account of the implications of idea that 'as well off' should be understood to mean that the present discounted value of current and future utility should be unchanged over the interval considered.

We work in the context of a general equilibrium taking the prices established in the general equilibrium as given. We first look at income from the production side showing that, in an economy where prices and the interest rates satisfy an intertemporal efficiency condition, income along an equilibrium path can be interpreted as a weighted average of the value of the current and future consumption bundle plus a term adjusting for changes in the nominal price level. Converting prices to 'real' prices by deflating by the Divisia consumption price index removes the need for this adjustment. These results are suggestive. First they offer a way of defining the income of a household entirely in terms of its consumption possibilities. Secondly they indicate that, to calculate real income, deflation is best carried out using a price index of consumption; this is a substantial divergence from current national accounting practice. Thirdly, because we have assumed constant returns, there is a trivial aggregation result that aggregate income is equal to the sum of the incomes from the individual sectors.

We explore these ideas in the household sector. For a single infinitely-lived household with an initial endowment, standard preferences and prices measured relative to its Divisia consumption price index, we define 'Real Income' as a weighted average of the value of its current and future consumption bundle . We then show that real saving, Real Income minus consumption, is directly proportional to the rate of change of welfare; implying that if consumption is equal to Real Income, then welfare is constant. This immediately suggests an interpretation of Hicks' original definition of income. If 'as well off' is understood as meaning having an unchanged sum of current and discounted future utility, then a household's income should be defined as equal to our Real Income measure.

We then proceed to show that this measure of income can be aggregated across multiple infinitely lived households with heterogeneous consumption preferences. We use existing results on general equilibrium with finitely many heterogeneous consumers to relate saving out of this aggregate measure of income to changes in an aggregate measure of social welfare. This avoids using the contrivance of a representative consumer. Further this measure of aggregate Real Income is identical to the income measure derived from the production side.

The question of capital gains can be explored by decomposing wealth into human capital (capitalised labour income) and claims on produced capital. In a society where net saving or net dissaving is taking place the wage rate and the rate of return, appropriately measured in real terms are changing, with the return to the increasingly scarce factor rising at the expense of the increasingly abundant factor of production (the Stolper-Samuelson theorem). We show that, at least in these circumstances, capital gains represent the capitalisation of the effects of these future factor price changes and are, effectively, a transfer from one factor to another rather than an increase in the resources available to the whole economy. For this reason capital gains should be included as part of a income only to the extent that the household or households stand to gain by changes in the relative prices of the factors. In a closed economy, as all factors are entirely owned by the households, the net gains are zero; thus, in a closed economy, capital gains should *not* be included in any income measure.

Our definition of income in a closed economy extends naturally to that of an open economy. Thus our general framework allows us to deal with the effects of changes in the international terms of trade on national income. It therefore encompasses naturally the solution Sefton & Weale (1996) offer to the 'Kuwait problem' - that if income is adjusted for resource depletion as is generally regarded appropriate in a closed economy- then the income of an economy which does nothing except produce exhaustible resources for export appears to be zero. We show, in the process, that the approach currently recommended in the United Nations System of National Accounts is not coherent with a general equilibrium definition of income. Finally we note that, in a Hecksher-Ohlin model where trade arises from international differences in factor endowments, there is a link between changes in the terms of trade and capital gains on net foreign investment, because a capital-rich economy has to be indifferent between producing capital intensive goods and employing its surplus capital abroad ; a measure of national income should similarly be unaffected by the arbitrary choice about the way in which excess capital is deployed. We use this relationship to suggest possible approaches to estimating the magnitude of the terms of trade effect in practice.

Throughout our analysis we assume, in addition to constant returns to scale, that labour is not traded between either households or countries, that all agents have the same discount rate and that output can simply be measured net of depreciation. The assumption of constant returns to scale is crucial to our results while the other assumptions serve only to simplify the analysis. We also look at an economy in which the government sector is not distinguished and, in particular, there are no distortionary indirect taxes.

1.1 Notation and Structure of the Paper

The structure of the paper is as follows. We consider a multi-sector production economy populated by a finite number of infinitely-live heterogeneous households. We assume the dynamics of this growth economy are described by an interior equilibrium; the Appendix discusses necessary and sufficient conditions for such an equilibrium to exist. The dynamic path for prices, therefore, satisfies the standard intertemporal efficiency and transversality conditions. Hence in the usual manner, we can split the economy into a production side, where firms maximise profits subject to a set of prices, and a demand side where households maximise their welfare subject to the same set of prices.

Now the problem of defining a measure of income can be understood as the problem of devising a rule for dividing out the returns to the factors of production between the households so as to convey information about the households' claim on current and future consumption resources and so, ultimately, their economic well-being. We therefore start our paper, Section

2, with a discussion of production in a closed economy. We show that under general conditions that net output of the sector is equal to a weighted sum of present and future consumption. In this way we reduce the problem of dealing with consumption and investment goods to one of dealing with a dynamic path for consumption goods. This suggests a concept of income that directly relates product to income. In section 3, we apply the concept of income to a single household and compare our definition of income with others in the literature. Section 4 discusses the issue of aggregating across households and provides a welfare interpretation of aggregate saving. In section 5 we consider the question of capital gains and in section 6 the open economy. Section 7 concludes.

Whilst we define each variable as we proceed, it is helpful to set out the notation here. On the production side we consider n_i firms, focusing on the ith firm denoted by the subscript i. These n_i firms produce at any time some output mixture of the n_j goods, indexed by the subscript j. We also consider n_k households and set out our arguments with reference to the kth household denoted by the subscript k. Variables which refer to the whole economy are not subscripted; these may be either the sum of the variables relating to individual firms (in the case of variables such as output) or some form of average, as in the case of the economy-wide real interest rates which need to be distinguished from the rates relating to particular households. The superscript R is used to denote variables in "real terms". These may be expenditure or prices deflated by an appropriate price index, or the rate of interest with an appropriate inflation rate subtracted from the nominal rate of interest. Most variables are functions of time. This is indicated explicitly only when it helps to clarify the argument.

2 Production in a Closed Economy

We consider a multi-sector neoclassical growth model. We index the sectors or firms by i and the goods by j. The n_j-vector of net output from firm i at a time t is denoted by $\mathbf{y}_i(t)$; thus $\mathbf{y}_{ij}$ refers the amount of good j produced by firm i. We make no distinction in the notation between outputs can be used as capital goods or consumed; this simplifies the analysis compared with a situation where capital goods are identified specifically. If $\mathbf{k}_i(t)$ denotes the n_j-vector of non-negative capital inputs and $l_i(t)$ denotes the non-negative input of labour, the technology of firm i is described by the production function

$$g_i\left(\mathbf{y}_i(t), \mathbf{k}_i(t), l_i(t)\right) = 0 \tag{1}$$

net of any depreciation. In the appendix we discuss possible assumptions concerning the technology that are sufficient to ensure an interior solution to the growth model. These assumptions are not so restrictive as to exclude all examples of production processes consuming at least one essential[1] exhaustible resource. However, the only assumption that is used directly in the derivation of our results is that the technology exhibits constant returns to scale (CRS), or that the production function g_i is homogeneous of degree zero. This assumption is necessary if we are to derive any strict accounting identities because only in this case are the returns to production exactly accounted for by the payments to the factors of production.

[1] An essential resource is one whose marginal product tends to ∞ as the amount consumed in production tends to 0. Dasgupta & Heal (1974) show that in single sector model this condition is sufficient to ensure that the resouce is never entirely depleted.

Each firm faces the same n_j-vector of prices $\mathbf{p}(t)$ for its outputs, the same n_j-vector of rental charges $\boldsymbol{\rho}(t)$ for its capital[2] and pays the same wage $w(t)$ for labour. If the nominal market rate of interest on cash is denoted $r(t)$, then the price and rental charge of capital good j and the interest rate along the equilibrium path are related by the no-arbitrage condition

$$rp_j - \dot{p}_j = \rho_j \qquad \text{or} \tag{2}$$

$$r - \frac{\dot{p}_j}{p_j} = \frac{\rho_j}{p_j}. \tag{3}$$

The path for prices is unique up to a scalar function of time. Thus if the prices $\mathbf{p}(t)$, $\boldsymbol{\rho}(t)$,$w(t)$ and interest rate $r(t)$ support the equilibrium then so do the prices $\mathbf{p}(t)/\lambda(t)$, $\boldsymbol{\rho}(t)/\lambda(t)$, $w(t)/\lambda(t)$ and interest rate $r(t) - \dot{\lambda}(t)/\lambda(t)$ where $\lambda(t)$ is a smooth scalar function of time. When we do not specify a particular normalisation for prices we refer to prices being in nominal or cash terms. Shortly we define a very specific normalisation and refer to prices under this normalisation as being measured in real terms. This is a convenient device to avoid having to specify the monetary side of our economy (and hence the process by which the price level is determined), yet still be able to talk about real and nominal prices[3].

The firm employs labour $l_i(t)$, and rents a capital stock $\mathbf{k}_i(t)$ so as to produce a vector of output $\mathbf{y}_i(t)$. It does so as to maximise the present discounted value of its profits

$$\int_t^\infty \left(\mathbf{p}(\tau)\mathbf{y}_i(\boldsymbol{\tau}) - w(\tau)l_i(\tau) - \boldsymbol{\rho}(\boldsymbol{\tau})\mathbf{k}_i(\tau)\right) e^{-\int_t^\tau r(v)dv} d\tau.$$

where the juxtaposition of two vectors always indicates their inner product. Under these assumptions, necessary conditions for an optimal production path require that the net marginal products always equal their respective prices.

For the moment we assume that the economy is closed. In this case, aggregate output from the firms, $\mathbf{y} = \sum_{i=1}^n \mathbf{y}_i$ is used either for consumption, $\mathbf{c}$, or to increase the aggregate capital stock, $\mathbf{k} = \sum_{i=1}^n \mathbf{k}_i$ implying the goods balance that $\mathbf{y} = \mathbf{c} + \dot{\mathbf{k}}$. Later we open the economy. Then some output could be traded, rather than either consumed or invested. Adjusting our expressions for Net National Product in the light of this trade is the focus of last section of the paper.

We can now identity a specific price normalisation with respect to a given consumption bundle. Under this normalisation, we require that the rate of price increase of the consumption bundle to be zero. We refer to prices under this normalisation as being real with respect to this

[2]If the goods are not used as capital goods in the production process, these rents are notional rents only. In the appendix, we consider two such types of goods; a consumption good and an exhaustible resource. In the first case, the associated capital stock is always zero and so in equilibrium the income from the associated factor is zero despite the notional rent being positive. In the latter case, the associated capital stock is positive but the notional rent is zero. So again the income from the associated capital factor is zero.

[3]We could have defined a 'nominal price' normalisation as a normalisation with respect to an arbitary time-varying bundle of goods, $\mathbf{b}(t)$. Given this bundle, prices are scaled in every period so that the bundle has a unit cost, $\mathbf{p}(t)\mathbf{b}(t) = 1$. The bundle $\mathbf{b}(t)$ can therefore be regarded as having a value equivalent to a unit of cash. As an example, define the bundle $\mathbf{b}(t)$ to be equal to a unit of good 1 divided by t. Then along the equilibrium path $\mathbf{p}_1 = t$. So the nominal price of good 1 measured in units of cash rises linearly with time. However this introduces significant extra notation for no gain.

consumption bundle. We now illustrate this price normalisation with respect to the aggregate consumption bundle. Thus let $\Pi(t)$ be the Divisia price index of the consumption bundle, $\mathbf{c}$. $\Pi(t)$ satisfies the differential equation

$$\dot{\Pi}(t)/\Pi(t) = \dot{\mathbf{p}}(\mathbf{t})\mathbf{c}(\mathbf{t})/\mathbf{p}(\mathbf{t})\mathbf{c}(t) \text{ and the initial condition } \Pi(0) = 1. \tag{4}$$

The initial condition is arbitrary, and is included only so as to define the Divisia index uniquely. 'Real' prices with respect to the aggregate consumption bundle $\mathbf{c}(t)$ are then equal to nominal prices deflated by the Divisia price index. These prices are denoted $\mathbf{p}^R(t) = \mathbf{p}(\mathbf{t})/\Pi(t)$, $\boldsymbol{\rho}^R(t) =$

$\boldsymbol{\rho}(\mathbf{t})/\Pi(t)$ and $w^R(t) = w(t)/\Pi(t)$. Therefore by construction the rate of real price inflation of the consumption bundle, $\dot{\mathbf{p}}^R(t)\mathbf{c}(\mathbf{t})/\mathbf{p}^R(t)\mathbf{c}(t)$, is zero. Further the real rate of interest, $r^R(t)$, consistent with this normalisation, is

$$r^R(t) = r(t) - \frac{\dot{\Pi}(t)}{\Pi(t)}. \tag{5}$$

In the next proposition, we proceed to show there is a direct relationship between Net National Product (NNP), conventionally measured as the net output of the production process, and present and future discounted levels of consumption.

Proposition 1 *Define prices as real with respect to aggregate consumption, and define real Net National Product, NNP^R as equal to the total real income or output from production,*

$$NNP^R(t) = \mathbf{p}^R(t)\mathbf{y}(t) = w^R(t)l(t) + \boldsymbol{\rho}^R(t)\mathbf{k}(t) \tag{6}$$

then real NNP is equal to a weighted present discounted sum of future consumption at all points ***along the equilibrium path***

$$NNP^R(t) = \int_t^\infty r^R(\tau)\mathbf{p}^R(\tau)\mathbf{c}(\tau)e^{-\int_t^\tau r^R(v)dv}d\tau \tag{7}$$

The proof is given in the appendix.

This proposition links closed-economy real NNP to the level of current and future real consumption; or more precisely to a weighted average of current and future consumption; the weights, $r^R(\tau)e^{-\int_t^\tau r^R(v)dv}d\tau$, integrate to 1 over the equilibrium path $\tau = t$ to ∞. In this way, national product is a measure of future consumption possibilities of a closed economy in a competitive equilibrium. It suggests an approach to relating the concepts of product and income, as links between production output to future consumption possibilities.

We now wish to make the following three remarks concerning this proposition..

1. The proposition is expressed in real terms. A similar expression can be derived in nominal terms. Integrating equation (5) implies that

$$\frac{\Pi(\tau)}{\Pi(t)}e^{-\int_t^\tau r(v)dv} = e^{-\int_t^\tau r^R(v)dv}$$

and substituting this and equation (5) into equation (7) and multiplying through by $\Pi(t)$ gives the following expression for NNP in nominal prices

$$NNP(t) = \mathbf{p}(t)\mathbf{y}(t) = \int_t^\infty r(\tau)\mathbf{p}(\tau)\mathbf{c}(\tau)e^{-\int_t^\tau r(v)dv}d\tau - \int_t^\infty \dot{\mathbf{p}}(\tau)\mathbf{c}(\tau)e^{-\int_t^\tau r(v)dv}d\tau. \quad (8)$$

All prices are now expressed in nominal units. This has introduced an additional term which is a present value sum of future changes in the price of the consumption bundle. This term is an adjustment to the weighted average of future consumption for changes in the price of the consumption bundle. Choosing to measure prices with respect to the consumption Divisia price index removes this term, thereby relating the level of production directly to a weighted average of future consumption possibilities.

2. The assumption of constant returns to scale (CRS) technology is required only to prove the equality in equation (6). It is not required to prove the identity in equation (7). This identity is derived using the first order conditions for a competitive equilibrium only.

3. Implicit within equation (6) is a trivial aggregation result; total Net National Product is the sum of the products of the n_i different firms or industries

$$NNP^R(t) = \sum_{i=1}^{n_i} \mathbf{p}^R(t)\mathbf{y}_i(t) = \sum_{i=1}^{n_i} w(t)l_i(t) + \sum_{i=1}^{n_i} \boldsymbol{\rho}^R(t)\mathbf{k}_i(t) \quad (9)$$

Aggregation on the production side is therefore a straightforward consequence of the CRS assumption. However later we derive a similar aggregation result for households.

Motivated by these results, we make the following definition.

Definition 2 *For a household k with future consumption* $\mathbf{c}_k(t)$*, define its Real Income,* $RI_k^R(t)$*, as a weighted average of its future consumption*

$$RI_k^R(t) = \int_t^\infty r_k^R(\tau)\mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau$$

where prices and interest rates are expressed in 'real' terms with respect to the consumption bundle $\mathbf{c}_k(t)$*, and are written as* $\mathbf{p}_k^R(t)$ *and* $r_k^R(t)$ *respectively. The deflator that converts nominal prices to real prices of household k is denoted* $\Pi_k(t)$*, where* $\mathbf{p}_k^R(t) = \mathbf{p}(t)/\Pi_k(t)$*.*

We have defined Real Income as a weighted average of current and future consumption, where the weights are functions of the instantaneous real interest rate. Later in Section 5, we show how this definition relates to the household's return to wealth and labour income.

3 Real Income and Welfare

Having looked at the relationship in a closed economy between output, NNP, and Real Income; we now turn to study the household sector so as to develop a link between Real Income and a household's welfare. More precisely, we consider the intertemporal partial equilibrium of a household, in order to show its savings flow, Real Income minus consumption, is proportional to the rate of change of its welfare. It is with this result that we are able to give income the following welfare interpretation; it is the maximum amount a household can consume while keeping the present discounted value of current and future utility constant.

We assume that household k derives utility $u_k(\mathbf{c}_k(t))$ from the consumption of goods $\mathbf{c}_k(t)$. Further this choice of consumption, $\mathbf{c}_k(t)$, maximises its welfare at time t, subject to its budget constraint. Household welfare is defined as the present value of the sum of current and future utility discounted at a constant rate θ,

$$\int_t^\infty u_k\left(\mathbf{c}_k(\tau)\right) e^{-\theta(\tau-t)} d\tau. \tag{10}$$

The budget constraint is that the present value of the household's future earnings plus any financial wealth, $h_k(t)$ equals the present value of its future consumption taking prices as given. In nominal terms, the budget constraint can be expressed as

$$\int_t^\infty \mathbf{p}(\boldsymbol{\tau})\mathbf{c}_k(\tau)e^{-\int_t^\tau r(v)dv} d\tau = h_k(t) + \int_t^\infty w(\tau)l_k e^{-\int_t^\tau r(v)dv} d\tau \tag{11}$$

where l_k denotes the fixed supply of labour of household k.

It will be useful to consider the associated indirect utility function representing the maximum attainable level of instantaneous utility given money expenditure m_k and a price vector $\mathbf{p}(t)$. Denote this function as $z_k(m_k, \mathbf{p}(t))$ then

$$z_k(m_k, \mathbf{p}(t)) = \max_{\widetilde{\mathbf{c}}} \; u_k(\widetilde{\mathbf{c}}) \qquad \text{given that} \quad \mathbf{p}(t)\widetilde{\mathbf{c}} \leq m_k. \tag{12}$$

If we define, money expenditure at time t to be equal to household expenditure at time t, $m_k(t) = \mathbf{p}(t)\mathbf{c}_k(t)$, then by definition the consumption bundle that achieves the maximum in expression (12) is the household consumption bundle $\mathbf{c}_k(t)$. The indirect utility function has the following well-known properties that

$$\frac{\partial z_k}{\partial m_k}\mathbf{p}(t) = \frac{\partial u_k}{\partial \mathbf{c}_k(t)} \qquad \text{and} \qquad \frac{\partial z_k}{\partial \mathbf{p}(t)} = -\frac{\partial z_k}{\partial m_k}\mathbf{c}_k(t) \tag{13}$$

where $\frac{\partial z_k}{\partial m_k}$ is referred to as the nominal marginal utility of money[4]. It converts nominal prices into units of marginal utility. As we frequently need to refer to this marginal utility of money at different points along the optimal consumption path, we abbreviate the notation to $z_k'(t)$. All these relationships could equally be expressed in real prices with respect to the consumption bundle $\mathbf{c}_k(t)$. In this case, we denote the marginal utility of money as $z_k^{R\prime}(t)$ and refer to it as the real marginal utility of money of household k. From the first property in equation (13), it

[4]Further the indirect utility function is concave and homogeneous of degree zero, $z_k(\lambda m_k, \lambda \mathbf{p}) = z_k(m_k, \mathbf{p})$.

is clear that the nominal marginal utility times the deflator equals the real marginal utility of money, that is

$$z_k^{R\prime}(t) = z_k^{\prime}(t)\Pi_k(t) \tag{14}$$

Now necessary conditions for an optimal or equilibrium consumption path are that prices are proportional to marginal utility and that the marginal utility of money satisfies the following intertemporal efficiency condition

$$r(t) = \theta - \frac{\frac{d}{dt}\left(z_k^{\prime}(t)\right)}{z_k^{\prime}(t)} \text{ or equivalently } r_k^R(t) = \theta - \frac{\frac{d}{dt}\left(z_k^{R\prime}(t)\right)}{z_k^{R\prime}(t)}. \tag{15}$$

In a manner similar to Proposition 1, we can use this first order to condition to link household saving to future changes in household welfare.

Proposition 3 *For household k with consumption path $\mathbf{c}_k(t)$, define Real Income and prices as in Definition 2. Then real savings, Real Income RI_k^R minus current real consumption, multiplied by the real marginal utility of money is equal to the rate of change of the household's welfare*

$$\frac{d}{dt}\left(\int_t^{\infty} u_k\left(\mathbf{c}_k(\tau)\right) e^{-\theta(\tau-t)} d\tau\right) = z_k^{R\prime}(t)\left(RI_k^R(t) - \mathbf{p}_k^R(t)\mathbf{c}_k(t)\right) \tag{16}$$

at all points along the equilibrium path.

Again the proof is deferred to the appendix. This proposition relates the rate of change of a household's welfare to its Real Income, i.e. a weighted average of their current and future consumption. As such it is a generalisation of the results in Sefton & Weale (1996), Weitzman & Asheim (2001) and Pemberton & Ulph (2001), all of whom relate the rate of change of a social welfare function to the level of net investment in a closed economy. In turn these results, can be seen as a generalisation of what has become known as Hartwick's Rule (Dixit, Hammond & Hoel 1980), that the social welfare function in a closed economy is constant if and only if net investment is zero. We shall discuss this result later, once we have derived the aggregation results.

We are now in a position to discuss our definition of Real Income. We first quote at length from Hicks' discussion of income in Hicks (1939, Chapter XIV) because it is both apposite and has been very influential.

> 'The purpose of income in practical affairs is to give people an indication of the amount which they can consume without impoverishing themselves. Following out this idea, it would seem that we ought to define a man's income as the maximum amount which he can consume during a week and still be as well off at the end of the week as he was at the beginning. Thus, when a person saves, he plans to be better off in the future; when he lives beyond his income, he plans to be worse off.' (Hicks 1939).

In determining whether a man is as well off at the end of the week, Hicks and others have addressed the question in terms of future achievable consumption levels; the above proposition suggests, however, that it is more sensibly addressed in terms of welfare. The result in proposition 3 demonstrates that, when a household spends its Real Income, its welfare is constant and thus it is indeed "as well off at the end of the week as it was at the beginning". This implies too, of course, that if it spends less than its Real Income it 'plans to be better off' and if it spends more it 'plans to be worse off' in a welfare sense. As such, our definition of real income offers a precise implementation of the Hicksian concept of income.

In the next subsection we compare our measure of income to others that have been suggested in the literature. However, this subsection can be omitted without losing the flow of the argument.

3.1 A comparison of our definition of Real Income with some alternatives

3.1.1 Income as the return on wealth

In his seminal paper, Weitzman (1976) defined income as the *stationary equivalent* of future consumption. In his analysis, he assumed a single composite consumption good; for the rest of this section, we assume the same. He defines income, which we denote as $WI_k^R(t)$, in his equation (10)[5] as,

$$WI_k^R(t) = r_k^R(t) \int_t^\infty \mathbf{c}_k(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau \tag{17}$$

where the real interest rate is defined relative to a consumption price of unity[6]. Given there is a single consumption good, the price normalisation is identical for every population, and so the superscript R and subscript k are redundant. We therefore omit them for the scope of this section. If real interest rates are constant $r(t) = r$ then equation (17) can be rewritten as

$$\int_t^\infty WI(t) e^{-r\tau} d\tau = \int_t^\infty \mathbf{c}(\tau) e^{-r\tau} d\tau \tag{18}$$

Thus the present value of the constant consumption path $WI(t)$ is equal the present value of consumption along a competitive trajectory with constant interest rates. It is for this reason, that Weitzman termed his definition of income as the *stationary equivalent* of future consumption. Asheim (1994) generalised Weitzman's concept of income as the *stationary equivalent* of future consumption to the case where interest rates are not constant. We discuss this generalisation later. For the moment, we follow Usher (1994) and refer to Weitzman's definition of income as the return to wealth, where the wealth of the economy is the present value of its future consumption[7]. In line with this we define consumption wealth as the integral on the right hand side of equation (17), and denote it as $CW(t)$ so that we can write $WI(t) = r(t)CW(t)$.

[5] It is of more than passing interest that integrating his equation (14) for the rate of change of income when interest rates are not constant gives our equation (7) for Real Income rather than his equation (10).

[6] This is consitent with equation (4) when there is a single consumption good.

[7] In private communication, Weitzman indicated that he is not comfortable with this interpretation. Rather he prefers to map the economy to an equivalent one, where the competive path is characterised by constant interest rates. Within this economy, his income measure has the interpretation as being the *stationary equivalent* of future consumption.

The mapping is to transform the consumption good $\mathbf{c}(t)$ to an alternative composite good $\widetilde{\mathbf{c}}(t) = f(\mathbf{c}(t))$.This economy is equivalent if the inverse mapping is applied both to the utility function and the production technology. Thus the utility function in the equivalent economy is $\widetilde{u}(x) = u(f^{-1}(x))$. Now the mapping, f, can be chosen

Now Hicks (1939), himself, pointed out the problem with any definition of income as the return to wealth. In point **4** of Chapter XIV he says 'consider what happens ... if interest rates are expected to change'. Though Hicks focuses on the return to money capital, the point is still valid.

In Section 5, we derive a relationship between Real Income and income as the return to wealth by integrating our definition of Real Income by parts[8]. For the moment we simply state the result,

$$RI(t) - WI(t) = \int_t^\infty \dot{r}(\tau)CW(\tau)e^{-\int_t^\tau r_k^R(\upsilon)d\upsilon}d\tau \tag{19}$$

The difference between the two measures is equal to a weighted sum of future changes in the interest rate. The weights are a function of the future levels of consumption wealth and the interest rate. Now if the interest rate is declining along the equilibrium path, as it will in any standard growth model with diminishing marginal returns to capital and an expanding capital stock, then $RI(t)$ will be less than $WI(t)$. In this case, Real Income is equal to the return to wealth plus an adjustment for the impact of declining interest rates on welfare. To understand the need for the adjustment, assume a household's consumption equalled its return to wealth then as the interest rate fell so would its consumption and therefore welfare. Therefore if income is to measure the amount that can be consumed whilst retaining welfare constant, it must be less than the return to wealth .

3.1.2 Income as the *stationary equivalent* of future consumption

Earlier, we noted that Asheim (1994) generalised Weitzman's concept of income as the *stationary equivalent* of future consumption to the case where interest rates are no longer constant. We denote his definition of income as $AI(t)$, where this measure is defined by the condition that

$$\int_t^\infty AI(t)e^{-\int_t^\tau r(\upsilon)d\upsilon}d\tau = \int_t^\infty \mathbf{c}(\tau)e^{-\int_t^\tau r(\upsilon)d\upsilon}d\tau. \tag{20}$$

Now as Asheim notes, the infinitely long term interest rate, $r_L(t)$ is defined as

$$r_L(t) = \frac{1}{\int_t^\infty e^{-\int_t^\tau r(\upsilon)d\upsilon}d\tau} \tag{21}$$

given the integral is bounded[9]. Therefore this definition of income can be rewritten as

$$AI(t) = r_L(t)\int_t^\infty \mathbf{c}(\tau)e^{-\int_t^\tau r(\upsilon)d\upsilon}d\tau \tag{22}$$

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so that interest rates are constant in this economy. However it is also possible to show (proof is available on request) that this must imply that the transformed utility $\widetilde{u}(x)$ function no longer has a postive but diminishing marginal utility. Further the technology no longer exhibits constant returns to scale. We therefore believe interpretation of the results in this economy is difficult.

For this reason, we shall continue to consider Weitzman's definition of income as the returns to wealth (as many have done so since) and note that Weitzman, himself, is more comfortable with the Asheim treatment that we shall discuss shortly.

[8] This identity is derived in the Section 5.

[9] For the exhaustible resource model discussed later in this section, this integral is sometimes unbounded. However we do not derive conditions for when this is the case, but merely sidestep the issue by assuming in these cases the long run interest rate is zero.

Thus, in an economic environment where interest rates vary, income as the *stationary equivalent* of future consumption is equal to the long run interest rate (rather than the instantaneous rate) times consumption wealth[10]. This definition of income is a literal implementation of Hicks' income definition No 3, 'the maximum amount a man can consume during a week and *every week thereafter* given his wealth at the beginning.' Asheim (1994) argues for this measure on the grounds of economic sustainability; it is the maximum sustainable level of consumption if consumption choices could be changed *without* changing supporting prices[11].

In the Appendix we derive the following relationship between Real Income and the income measure[12], $AI(t)$:

$$RI(t) - AI(t) = \int_t^{\infty} \left(1 - \frac{r_L(t)}{r_L(\tau)}\right) \dot{\mathbf{c}}(\tau) e^{-\int_t^{\tau} r(v)dv} d\tau. \tag{23}$$

In this case the difference between the two income measures is a weighted sum of current and future consumption growth rates. The weights are a function of the long run and instantaneous interest rates. Again let us consider the case when instantaneous interest rates and, therefore long run rates too, are declining along the equilibrium path. Along this path, aggregate consumption growth is positive as the capital stock is expanding and therefore the 'representative' household will also experience growing consumption. Hence, $\dot{\mathbf{c}}(\tau) > 0$, and the the term $\left(1 - \frac{r_L(t)}{r_L(\tau)}\right)$ is negative for $\tau > t$ as interest rates are declining, implying that the integral on the right hand side of equation (23) is negative. Therefore, for this 'representative' household, Real Income is always less than income as the *stationary equivalent* of future consumption. The intuition behind this result is straightforward. The income measure, $AI(t)$, is equal to the constant level of consumption that has the same present value as the household's consumption path in equilibrium. If consumption is always increasing along this path, this weighted average must be greater than current consumption.

This simple example suggests a difficulty with interpreting this income measure as the maximum sustainable level of consumption. For if our 'representative' household actually consumed an amount equal to this measure of income, their savings would be negative and so their consumption levels must eventually fall; suggesting this measure of income is far from a *sustainable* level of consumption. Asheim (1994) was aware of this difficulty for as he notes, this income measure is only equal to maximum level of attainable consumption if the competitive prices support a constant consumption level otherwise it simply an upper bound. One could argue

[10] In the appendix, we also show that this income measure can also be interpretated as a weighted average of current and future consumption. For this income measure can be rewritten as

$$AI(t) = \int_t^{\infty} r_L(\tau)\mathbf{c}(\tau) e^{-\int_t^{\tau} r_L(v)dv} d\tau$$

where the weights, this time, are a function of the long run interest rate and again sum to 1.

[11] This can be contrasted to Solow's (1974) treatment. Solow argues that an extreme interpretation of the Rawlsian savings principle for intergenerational equity is for parents to only to consume a level of consumption that is sustainable indefinitely; he therefore calculates the maximum attainable consumption level that could be supported by a set of competitive prices.

[12] This relation uses the third identity proved in Lemma 7, and a similar manipulation on the measure $AI(t)$ which is described in the Appendix.

that Real Income is a better measure of sustainability, but a measure of the sustainability of welfare rather than consumption.

3.1.3 Income as a Hamiltonian

Finally we discuss a number of papers that define income as a Hamiltonian, e.g. Usher (1994), Weitzman (2000), Hartwick (1990) and Pemberton & Ulph (2001). Their approaches are related to our equilibrium approach, though the link is far from transparent. We try to briefly summarise the connection here. First we must rewrite the rate of change of welfare as equal to the discount rate, θ, times current welfare minus current utility, that is

$$\frac{d}{dt}\left(\int_t^\infty u_k\left(\mathbf{c}_k(\tau)\right)e^{-\theta(\tau-t)}d\tau\right)=\int_t^\infty \theta u_k\left(\mathbf{c}_k(\tau)\right)e^{-\theta(\tau-t)}d\tau - u_k\left(\mathbf{c}_k(t)\right) \tag{24}$$

In a simple centrally planned growth model, the first term on the right hand side, which we call welfare income, is equal to the value of the Hamiltonian of the associated optimisation problem. With an alarming lack of rigour, we write the Hamiltonian, $\mathcal{H}(q,k)$, to the associated optimisation problem as

$$\mathcal{H}(q,k)=u_k\left(\mathbf{c}_k(t)\right)+\mathbf{q}(t)\dot{\mathbf{k}}(t) \tag{25}$$

where $q(t)$ are the Lagrange multipliers or shadow prices, $\mathbf{k}\,(t)$ are the states of the problem and $\mathbf{c}_k(t)$ is the optimal consumption level that maximises the Hamiltonian. Now we can substitute out for utility in terms of the Hamiltonian in the expression for welfare income,

$$\int_t^\infty \theta u_k\left(\mathbf{c}_k(\tau)\right)e^{-\theta(\tau-t)}d\tau=\int_t^\infty \theta\left(\mathcal{H}(q(\tau),k(\tau))-\mathbf{q}(t)\dot{\mathbf{k}}(t)\right)e^{-\theta(\tau-t)}d\tau \tag{26}$$

and integrate the first term on the right hand side by parts

$$\int_t^\infty \theta u_k\left(\mathbf{c}_k(\tau)\right)e^{-\theta(\tau-t)}d\tau=\mathcal{H}(q(t),k(t))+\int_t^\infty\left(\frac{d}{d\tau}\left(\mathcal{H}(q(\tau),k(\tau))\right)-\theta\mathbf{q}(t)\dot{\mathbf{k}}(t)\right)e^{-\theta(\tau-t)}d\tau. \tag{27}$$

where we have used a transversality condition to evaluate the Hamiltonian at ∞. The second term in the above equation is zero. This is a direct result of the dynamic equations describing the path of the Hamiltonian along the optimal trajectory,

$$\frac{\partial\mathcal{H}(q,k)}{\partial\mathbf{k}}=\theta\mathbf{q}(t)-\dot{\mathbf{q}}(t)\quad\text{and}\quad\frac{\partial\mathcal{H}(q,k)}{\partial\mathbf{q}}=\dot{\mathbf{k}}(t).$$

Thus we have established, if somewhat sketchily, that welfare income is equal to the Hamiltonian of the associated optimal planning problem. Further welfare income, from equation (24), is intimately connected to the rate of change of aggregate welfare and so the problems are intimately related. There remains the issue that the Hamiltonian is measured in utility units, whereas consumption and investment are measured units of money. The majority of authors, take a linear approximation of the utility function to convert from one to the other. However from equation (24) and equation (25), it is apparent that the rate of change of welfare is equal to $\mathbf{q}(t)\dot{\mathbf{k}}(t)$ and therefore the approximation is unnecessary. Despite the strong connection, linking income to the value of a Hamiltonian does little to illuminate the problem; it is a diversion, though an intriguing one.

4 Aggregate Saving and Welfare

The previous section considered the welfare of a single household in our competitive economy. In this section, we derive our aggregation results. We first show that the sum of each household's Real Income is equal to aggregate Real Income even if they have heterogeneous preferences. We also show that the sum of the aggregate real savings of the individual households is proportional to the rate of change of a social welfare function. This social welfare function is a weighted sum of the individual welfare functions and the constant of proportionality is the respective aggregate marginal utility of money. This result clarifies the link between social income and the individual household's income.

The first result follows in a straightfoward manner from our earlier definitions and is proved in the Appendix.

Proposition 4 *Define aggregate Real Income, $RI^R(t)$, as the weighted present discounted sum of aggregate future consumption*

$$RI^R(t) = \int_t^{\infty} r^R(\tau)\mathbf{p}^R(\tau)\mathbf{c}(\tau)e^{-\int_t^{\tau} r^R(v)dv}d\tau$$

where prices and interest rates are expressed in 'real' terms with respect to the aggregate consumption bundle $\mathbf{c}(t)$ and deflator $\Pi(t)$. Now real aggregate consumption and interest rates are weighted sums of the respective household quantities

$$\mathbf{p}^R(t)\mathbf{c}(t) = \sum_k \frac{\Pi_k(t)}{\Pi(t)}\mathbf{p}_k^R(t)\mathbf{c}_k(t) \qquad \text{and} \qquad r^R(t) = \sum_k \frac{\mathbf{p}(t)\mathbf{c}_k(t)}{\mathbf{p}(t)\mathbf{c}(t)} r_k^R(t)$$

and therefore so is Real Income

$$RI^R(t) = \sum_k \frac{\Pi_k(t)}{\Pi(t)} RI_k^R(t).$$

Thus aggregate Real income is simply the deflated sum of the individual household's real income, and the aggregate real interest rate is a *consumption-weighted* average of the individual household real interest rates.

The second result is a direct consequence of the Second Welfare Theorem. We have assumed that markets are competitive, and both technology and preferences are convex and so our resulting equilibrium is Pareto efficient. One can then consider a set of related Pareto efficient equilibrium. These equilibria are generated by maximising a social welfare function, defined as the weighted sum of our individual welfare functions, subject to the constraints implied by the technology and initial endowments. The first order conditions for maxima of these social welfare functions will be identical to the intertemporal efficiency conditions for a competitive equilibrium in our economy. Now the Second Welfare Theorem implies that all these equilibrium can be decentralised or supported with the use of lump sum transfer payments. We therefore need only to find the particular social welfare function that is supported by zero transfer payments. This is our aggregate welfare function. This approach to analysing competitive equilibria was first suggested by Negishi (1960) and generalised to dynamic models by Bewley

(1980). Kehoe, Levine & Romer (1990) then used this approach to show the determinacy of competitive equilibria in dynamic models with finitely many consumers. This is equivalent to showing the local uniqueness of the social welfare function.

Denote the social utility function by

$$u(\mathbf{c}) = \max\left(\sum_k \alpha_k u_k(\mathbf{c}_k): \quad \text{given that } \sum_k \mathbf{c}_k = \mathbf{c}\right) \tag{28}$$

where $\alpha_k \in \Re^+$. Lemma 1 of Kehoe et al. (1990) proves that this social welfare function is well-behaved if the individual utility functions are well-behaved. The weights α_k are chosen to be solution to the following finite dimensional Negishi problem. Let the consumption path, $\mathbf{c}(t)$, maximise the social welfare function at time t,

$$\int_t^\infty u\left(\mathbf{c}(\tau)\right) e^{-\theta(\tau-t)} d\tau \tag{29}$$

subject to the aggregate budget constraint

$$\int_t^\infty \mathbf{p}(\boldsymbol{\tau})\mathbf{c}(\tau)e^{-\int_t^\tau r(v)dv} d\tau = h(t) + \int_t^\infty w(\tau) l e^{-\int_t^\tau r(v)dv} d\tau \tag{30}$$

where aggregate wealth, $h(t)$, and aggregate labour supply , $l(t)$ are defined naturally as $h(t) = \sum_k h_k(t)$ and $l(t) = \sum_k l_k(t)$ respectively. Then the weights are chosen so as to ensure that the following individual budget constraints

$$\int_t^\infty \mathbf{p}(\boldsymbol{\tau})\mathbf{c}_k(\tau)e^{-\int_t^\tau r(v)dv} d\tau = h_k(t) + \int_t^\infty w(\tau) l_k e^{-\int_t^\tau r(v)dv} d\tau \tag{31}$$

are satisfied too. Even though there are k household budget equations there are only $k-1$ constraints. For if $k-1$ household budget equations are satisfied, then so is the k^{th} because of the aggregate budget constraint. Similarly there are only $k-1$ degrees of freedom in the choice of the weights as they are unique only unique up to a scalar. Kehoe et al. (1990) show that given the standard convexity assumptions, there is a locally unique solution to this problem.

In a manner identical to the analysis of the individual household, we define the aggregate indirect utility function, $z(m, \mathbf{p}(t))$, as

$$z(m, \mathbf{p}(t)) = \max_{\widetilde{\mathbf{c}}} \; u(\widetilde{\mathbf{c}}) \qquad \text{given that} \quad \mathbf{p}(t)\widetilde{\mathbf{c}} \leq m. \tag{32}$$

where expenditure, $m(t)$, at time t is set equal to $\mathbf{p}(t)\mathbf{c}(t)$. It also satisfies properties (13) *ceteris paribus*. We refer to $\frac{\partial z}{\partial m}$ as the aggregate marginal utility of money, and use the abbreviations $z'(t)$ and $z^{R\prime}(t)$ to refer to this marginal utility depending on whether prices are in nominal or real terms with respect to aggregate consumption bundle. Finally, given the first order conditions implicit in the maximisation (28), the aggregate marginal utility of money is related by the weights α_k to the household's marginal utility of money along the competitive path,

$$z'(t) = \alpha_k z_k'(t) \tag{33}$$

It is this relation that provides the basis for the following aggregation result.

Proposition 5 *Given the definitions in Proposition 4, real aggregate savings, aggregate Real Income RI^R minus aggregate real consumption, is related to the rate of change of the social welfare function by the aggregate marginal utility*

$$\frac{d}{dt}\left(\int_t^{\infty} u\left(\mathbf{c}(\tau)\right) e^{-\theta(\tau-t)} d\tau\right) = z^{RI}\left(RI^R(t) - \mathbf{p}^R(t)\mathbf{c}(t)\right) \tag{34}$$

at all points along the equilibrium path.

This proposition derives an aggregation result for the income side analogous to the aggregation result for the output side in Remark 3, page 6. The aggregation result uses the implicit weights implied by the Second Welfare Theorem to combine the individual welfare functions. As such, it articulates the relationship between social income and the individual household incomes.

A corollary of Propositions 1 and 5 is Theorem 1 of Dixit et al. (1980). This theorem is more commonly known as Hartwick's rule.

Corollary 6 *For a competitive closed economy, social welfare is constant if and only if aggregate net investment is zero.*

Proof Given Proposition 5, social welfare is constant if and only if aggregate Real Income is equal to aggregate real consumption. For a closed competitive economy, Proposition 1 proved that aggregate Real Income is equal to aggregate real consumption plus aggregate net investment. Therefore social welfare is constant if and only if aggregate net investment is zero. ■

5 Income, Wealth and Capital Gains

Eisner (1988*b*), in his detailed discussion of Extended National Income Accounts, argues that real capital gains should be included as part of income

> 'If we accept the Haig-Hicks-Simon definition of income as that which can be consumed while keeping real wealth intact, savings is the difference between this measure of income and actual consumption . Both income and saving will then include capital gains.'

In section 3.1 we argued for the definition of income as that which can be consumed while keeping welfare intact. This has the implication that capital gains should *not* be included as part of income in a closed economy at least in as far they arise from the process of capital accumulation and the resultant changes in real interest rates. We show here that these capital gains are equal to the future changes in consumption possibilities resulting from changes in the real interest rate. Therefore, by subtracting capital gains from the instantaneous rate of return to wealth, we adjust the return for the impact of changes in the interest rate on future consumption possibilities. This implies that only the rents from capital should be included as part of income.

We start off by stating a number of identities linking income and wealth which will be used throughout this section.

Lemma 7 *Define consumption wealth as the present value of future consumption expenditures or equivalently (from the intertemporal budget constraint (31)) as the sum of financial plus human capital*

$$\begin{aligned} CW_k^R(t) &= h_k(t) + \int_t^\infty w(\tau) l_k e^{-\int_t^\tau r(v)dv} d\tau \\ &= \int_t^\infty \mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau. \end{aligned}$$

Then Real Income of household k, as defined in Definition 2, can be written as the return to wealth plus terms of trade effects,

$$\begin{aligned} RI_k^R(t) &= r_k^R(t) CW_k^R(t) + \int_t^\infty \dot{r}_k^R(\tau) CW_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau && (35) \\ &= r_k^R(t) h_k(t) + w_k^R(t) l_k + \int_t^\infty \dot{r}_k^R(\tau) h_k(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau + && (36) \\ &\quad \int_t^\infty \dot{w}_k^R(\tau) l_k e^{-\int_t^\tau r_k^R(v)dv} d\tau \end{aligned}$$

or as current consumption plus the present value of future changes to consumption

$$RI_k^R(t) = \mathbf{p}_k^R(t)\mathbf{c}_k(t) + \int_t^\infty \mathbf{p}_k^R(\tau)\dot{\mathbf{c}}_k(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau. \quad (37)$$

All the identities are proved in the Appendix and involve integrating the definition for Real Income by parts.

The first identity states that Real Income is equal to the return to total wealth plus a term quantifying the effect of future changes in the interest rate on household welfare. This term is the weighted present value of all future changes in the intertemporal terms of trade where the weights are future total wealth holdings. Therefore Real Income equals the return to wealth only if this term is zero. This is tantamount to requiring that the economy be in a steady state.

In this paper we have treated the income from labour as the income on a resource rather than a return to a stock of human capital. The second identity therefore decomposes total wealth into financial wealth and human capital and re-expresses the income from human capital as labour income plus a term describing the impact of future changes in the wage rate on the household's welfare. There are now two terms of trade effects. We call the weighted present value of all future changes in the intertemporal terms of trade 'the intertemporal terms of trade effect' and the present value of all future changes in the labour terms of trade the 'labour terms of trade effect'. These terms of trade effects describe the impact of changes in the interest and wage rate on future consumption possibilities. We can quantify this relationship precisely by using the third identity which relates Real Income to current consumption plus the present value of future changes in consumption. By subtracting the third identity from the second, we derive the following equation

$$\begin{aligned} \left(r_k^R(t) h_k(t) + w_k^R(t) l_k \right. &- \left. \mathbf{p}_k^R(t)\mathbf{c}_k(t)\right) = \left(\int_t^\infty \dot{r}_k^R(\tau) h_k(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau \right. && (38) \\ &\left. + \int_t^\infty \dot{w}_k^R(\tau) l_k e^{-\int_t^\tau r_k^R(v)dv} d\tau - \int_t^\infty \mathbf{p}_k^R(\tau)\dot{\mathbf{c}}_k(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau \right) \end{aligned}$$

So if current consumption equals the return to financial wealth plus earnings, the intertemporal and labour terms of trade effects equal the present value of future changes to consumption.

These two terms of trade effects are intimately connected to the economy's capital gains. We now look at the case when household k refers to the population of a closed economy. In the next section we look at the more general case of an open economy. In a closed economy, the financial wealth of household k is equal to real value of the economy's capital stock, household k's labour supply is equal to the economy's labour supply and from Proposition 1 their Real Income is equal to the Net National Product. Thus,

$$\begin{aligned} RI_k^R(t) &= \boldsymbol{\rho}_k^R(t)\mathbf{k} + w_k^R(t)l_k \\ &= r_k^R(t)h_k(t) - \dot{\mathbf{p}}_k^R(t)\mathbf{k} + w_k^R(t)l_k \end{aligned}$$

where the second line follows from the intertemporal efficiency conditions (2). Therefore by equating this expression for Real Income with identity (36), we prove that in a closed economy the sum of the two terms of trade effects is equal to minus the real capital gains,

$$-\dot{\mathbf{p}}_k^R(t)\mathbf{k} = \int_t^\infty \dot{r}_k^R(\tau)h_k(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau + \int_t^\infty \dot{w}_k^R(\tau)l_k e^{-\int_t^\tau r_k^R(v)dv}d\tau \tag{39}$$

Equation (39) relates the changes in price of the outputs of a firm or sector to the changes in the prices of the inputs. We shall refer to equations of this form as a terms of trade balance.

We are now in a position to put our criticism of Eisner's work more explicitly. Real Income, as defined in Definition 2, is intimately linked to the rate of change of the household's welfare without the inclusion of capital gains. The inclusion of capital gains, which is related to future factor price movements, breaks this link and will result in a form of double-counting. It follows that although Eisner's proposal may be appropriate for capital gains arising in circumstances outside the scope of this model, it is not universally applicable.

6 The Open Economy and Terms of Trade

In this last section, we extend our results for a closed economy to an open economy. As we have shown in section 4 it is possible to aggregate up households. This makes it possible to use countries as our basic demand unit in this section, where each country should be regarded as made up of a finite number of households. We index these countries by k too, because there is no need to make any distinction between households and countries. In a closed economy all the goods produced, and not invested, are consumed, whereas in an open economy, these goods could also be traded. If the relative price change of the traded goods is different from the relative price change of the country's consumption bundle then this introduces a further terms of trade effect. In this section we quantify precisely this terms of trade effect. As in the previous section, we are able to relate this terms of trade effect to the capital gains accruing on the capital goods of a particular portfolio of industries.

In national accounting, national income is estimated by estimating the national product and adjusting it for net foreign property income. This is the approach adopted here first. Accordingly, associate the output of firm i with country k. This association is made explicit by assuming that all the labour for firm i is supplied by country k, that is $l_i = l_k$ so that capital, and not labour can be exported. We denote the vector of net exports by $\mathbf{x}_i(t)$ so that the open-economy goods balance can be written

$$\mathbf{y}_i(t)= \mathbf{c}_k(t)+\dot{\mathbf{k}}_i(t)+\mathbf{x}_i(t). \tag{40}$$

Country k's financial wealth can be held at home or abroad. We write this as

$$h_k(t) = \mathbf{p}_k^R(t)\mathbf{k}_i(t) + h_k^f(t) \tag{41}$$

where $h_k^f(t)$ is the value of net foreign assets. The introduction of trade introduces an extra term into the terms of trade balance in equation (39). This is because relative prices changes in the traded bundle are not necessarily equal to relative price changes in the consumption bundle. If the consumption bundle becomes cheaper (more expensive) relative to the traded bundle this introduces a positive (negative) terms of trade effect. This positive effect increases the country welfare; the terms of trade effect is a quantification of this welfare gain. Alternatively this effect can be related to the production sector. The real interest rate is defined such that the price of the consumption bundle is constant. In a closed economy this implies the price of the consumption goods produced by the production sector is constant too. This removes a term from the terms of trade balance.

The terms of trade balance of firm i in terms of the real interest of country k can be written[13]

$$\begin{aligned}\int_t^\infty \dot{\mathbf{p}}_k^R(\tau)\mathbf{x}_i(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau - \dot{\mathbf{p}}_k^R(t)\mathbf{k}_i &= \int_t^\infty \dot{r}_k^R(\tau)\mathbf{p}_k^R(\tau)\mathbf{k}_i(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau \\ &\quad + \int_t^\infty \dot{w}_k^R(\tau)l_k e^{-\int_t^\tau r_k^R(v)dv}d\tau \end{aligned} \tag{42}$$

This relationship can be substituted into equation (36) to derive an expression relating the Real Income of country k to the income from firm i,

$$\begin{aligned} RI_k^R(t) &= \boldsymbol{\rho}_k^R(t)\mathbf{k}_i(t) + w_k^R(t)l_k + r_k^R(t)h_k^f(t) + \\ &\quad \int_t^\infty \dot{r}_k^R(\tau)h_k^f(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau + \int_t^\infty \dot{\mathbf{p}}_k^R(\tau)\mathbf{x}_i(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau. \end{aligned} \tag{43}$$

As $l_i = l_k$ by assumption, the first two terms are equal to the income from firm i or Net National Product. The third term is the net property income from foreign assets. The next term is the intertemporal terms of trade effect on net foreign assets and the final term we call the terms of trade effect. Thus the Real Income of country k is equal to Net National Product plus net property income plus two terms of trade effects. These terms account for the impact of future changes in the returns to net foreign assets and future changes to the terms of trade of future consumption possibilities and therefore welfare.

6.1 An Example: The income of an exhaustible resource exporter

Sefton & Weale (1996) examine the special case of the National Income of a exporter of an exhaustible natural resource. We simplify the analysis here to illustrate the importance of the

[13] This is proved in Proposition 8 later.

terms of trade effect, and how its magnitude might be estimated in a special case. In this simplified example, the country has no assets except for a stock of the exhaustible resource. There is no production sector; the country simply exports the resource and imports the consumption good[14]. Since Usher (1994), this example has become to be known as the 'Kuwait oil problem'. Usher (1994) argues that if the stock of capital is treated as a capital good in line with the System of Environmental and Economic Accounting (Baretelmus, Stahmer & van Tongereren 1991) then the national income of this idealised country is zero for the value of consumption equals the value of negative investment in the resource. For a closed economy such a treatment is consistent with our earlier analysis and definition of income. However in an open economy this fails to take account of the terms of trade effect.

In a competitive world, the money price of an exhaustible resource (given no new discoveries) will rise at the interest rate; for otherwise, as Hotelling (1931) argued, the owners of the asset would be better off selling the entire asset stock and investing the proceeds. Denote the stock of the resource left at time t by $k_r(t)$, the amount extracted as $-\dot{k}_r(t)$ and the price of the resource as $p_r(t)$. From Hotelling's rule, the price of the resource at any time $\tau \geq t$ is

$$p_r(\tau) = p_r(t)e^{\int_t^\tau r(\upsilon)d\upsilon}.$$

We can now evaluate the Real Income of 'Kuwait'. The first three terms in equation (43) are zero, as there is no production in this economy (only a running down of the capital stock) and at time t there are no foreign assets. So the only non zero terms are the two terms of trade effects. For simplicity assume there is a single consumption good, and so the only changing price is the price of the resource relative to the consumption good.. Therefore the Real Income of 'Kuwait' is equal the last two terms of equation (43) which are

$$\begin{aligned} RI_k^R(t) &= \int_t^\infty \dot{r}_k^R(\tau)h_k^f(\tau)e^{-\int_t^\tau r_k^R(\upsilon)d\upsilon}d\tau + \int_t^\infty \frac{d}{d\tau}\left(p_r(t)e^{\int_t^\tau r(\upsilon)d\upsilon}\right)\left(-\dot{k}_r(\tau)\right)e^{-\int_t^\tau r_k^R(\upsilon)d\upsilon}d\tau. \\ &= \int_t^\infty \dot{r}_k^R(\tau)h_k^f(\tau)e^{-\int_t^\tau r_k^R(\upsilon)d\upsilon}d\tau - p_r(t)\int_t^\infty r(\tau)\dot{k}_r(\tau)d\tau. \qquad (44) \end{aligned}$$

This second term in this last line can now be integrated by parts to derive the following expression for Real Income

$$RI_k^R(t) = \int_t^\infty \dot{r}_k^R(\tau)\left(h_k^f(\tau) + k_r(\tau)\right)e^{-\int_t^\tau r_k^R(\upsilon)d\upsilon}d\tau + r(t)p_r(t)k_r(t). \qquad (45)$$

The first term is the intertemporal terms of trade effect on total assets, which now include the stock of the resource. The second term is the terms of trade effect, due to the rising price of the resource. This is equal to an imputed rent on the resource stock. Therefore the income of 'Kuwait' is what it would be if the stock of the resource were sold off and the proceeds invested with 'Kuwait's' income being the return to the invested assets plus an intertemporal terms of trade effect. Real Income of 'Kuwait' could be estimated from this expression by estimating the magnitude of the imputed rent from estimates of the remaining stock of the resource; and in the absence of any information to the contrary, the intertemporal terms of trade effects could be ignored.

[14] Sefton and Weale (1996) include a production sector, but this does not materially effect the issues.

6.2 Relating Real Income to the Production Sector in an Open Economy

Real Income, as defined in Definition 2, is a weighted average of current and future consumption. Though Real Income is defined in terms of current and future variables, we have shown that for a closed *competitive* economy Real Income is equal to the economy's net product. Thus it can be estimated from the production sector in terms of currently observable quantities. Alternatively one could understand Real Income as equal to current receipts (that is the return to wealth plus labour income) plus two terms of trade effects. These terms of trade effects account for the impact of future changes in the interest rate and the wage rate on future consumption possibilities. In a closed economy, these terms of trade effects are equal to minus the net real capital gains; see equation (39). So Real Income can be estimated as current income minus capital gains which is equal to the net product. Either way, one is able to estimate Real Income directly from current production.

Unfortunately these relationships no longer apply directly in an open economy. In this section we show it is possible to construct a set of industry portfolios such that each country's Real Income is equal to the output or product of their respective industry portfolio. Further these portfolios satisfy the aggregation property; that aggregate Real Income of a number of countries is just the sum of the product of their respective industry portfolios.

This result follows relatively easily from our earlier results. However we need one further identity. This identity is a generalisation of the terms of trade balance stated in equation (39) for an arbitrary industry portfolio. Denote the industry portfolio of country k at time t as $\mathbf{s}_k(t)$ where $\mathbf{s}_k(t)$ is vector of length n_i (the number of firms). If the elements of $\mathbf{s}_k(t)$ are denoted $s_{ik}(t)$ the output of this industry portfolio is $\sum_i s_{ik}(t)\mathbf{y}_i(t)$ and is denoted $\mathbf{y}_k(t)$. Thus we indicate that the output vector refers to an industry portfolio rather than the output vector of a firm by using a subscript k rather than a subscript i. It will be useful to use a matrix notation too. If we denote as $\mathbf{Y}(t)$ the concatenated matrix of the outputs of each firm at time t; that is $\mathbf{Y} = [\mathbf{y}_1\, \mathbf{y}_2\, \cdots\, \mathbf{y}_i\, \cdots \dot{]}$, then the output vector of industry portfolio k can be written as $\mathbf{y}_k(t) = \mathbf{Y}(t)\mathbf{s}_k(t)$. Further the value of this output at prices $\mathbf{p}(t)$ is just $\mathbf{p}(t)\mathbf{Y}(t)\mathbf{s}_k(t)$, where the vector $\mathbf{p}$ has been transposed to ensure valid multiplication, i.e. . $\mathbf{pY} = [\mathbf{py}_1\, \mathbf{py}_2\, \cdots \mathbf{py}_i\, \cdots \dot{]}$. Similarly denote by $\dot{\mathbf{k}}_k(t)$, $\mathbf{k}_k(t)$ and $\widetilde{l}_k(t)$ the investment in, the capital stock of and labour supply[15] to industry portfolio k. In matrix notation, these vectors of outputs and inputs to the industry portfolio k can be written naturally as $\dot{\mathbf{k}}_k(t) = \dot{\mathbf{K}}(t)\mathbf{s}_k(t)$, $\mathbf{k}_k(t) = \mathbf{K}(t)\mathbf{s}_k(t)$ and $\widetilde{l}_k(t) = \mathbf{l}(t)\mathbf{s}_k(t)$. We can now state the terms of trade balance for an arbitrary industry portfolio in terms of nominal prices.

Lemma 8 *Given an industry portfolio* $\mathbf{s}_k(t)$ *of country* k*, define the country's trade* $\mathbf{x}_k(t)$ *with respect to this portfolio by the goods balance*

$$\mathbf{y}_k(t) = \mathbf{c}_k(t) + \dot{\mathbf{k}}_k(t) + \mathbf{x}_k(t) \tag{46}$$

[15] The tilde on labour supply is neccesary to avoid ambiguity between the labour supply to industry portfolio k and the labour supply of household k. Later we shall require that these two quantities are equal, so removing any possible ambiguity.

then the terms of trade balance associated with this industry portfolio is

$$\int_t^\infty \dot{\mathbf{p}}(\tau)\left(\mathbf{x}_k(\tau)+\mathbf{c}_k(\tau)\right)e^{-\int_t^\tau r(v)dv}d\tau = \int_t^\infty \dot{r}(\tau)\mathbf{p}(\tau)\mathbf{k}_k(\tau)e^{-\int_t^\tau r(v)dv}d\tau + \int_t^\infty \dot{w}(\tau)\widetilde{l}_k e^{-\int_t^\tau r(v)dv}d\tau + \dot{\mathbf{p}}(t)\mathbf{k}_k \tag{47}$$

The result, proved in the Appendix, is dependent on the assumption that each firm exhibits constant to returns to scale and prices satisfy the intertemporal efficiency condition (2). So far we have used two special cases of this result. The first is when we are dealing with a closed economy, in which case the term involving trade, $\mathbf{x}_k(t)$, is zero; the second is when prices are defined in real times with respect to the consumption bundle of country k, in which case the term involving changes in the price of the consumption bundle, $\dot{\mathbf{p}}(t)\mathbf{c}_k(t)$, is zero. Clearly if both conditions are satisfied then both terms are zero.

The problem of estimating country k's Real Income can be reduced to the problem of finding an industry portfolio such that the terms of trade effects are zero with respect to this industry portfolio. There are three terms of trade effect; an intertemporal, a labour and an external trade effect. The portfolio must therefore satisfy three constraints. The first is the same as in the previous section, that total labour supplied to the industry is equal to country labour supply, $l_k(t) = \widetilde{l}_k(t) = \mathbf{l}(t)\mathbf{s}_k(t)$, so removing the labour terms of trade effect with respect to this industry portfolio. The second is that all the country's wealth is invested in the industry portfolio, $h_k(t) = \mathbf{p}(t)\mathbf{K}(t)\mathbf{s}_k(t)$, so that the intertemporal terms of trade is also removed. The third constraint removes the external terms of trade effect. For this we require that $\dot{\mathbf{p}}_k^R(t)\mathbf{x}_k(t) = 0$ for all t [16]. As we shall show in the proof to the following Lemma in the Appendix, this constraint can be reexpressed in nominal prices as $\dot{\mathbf{p}}(t)\mathbf{x}_k(t) = 0$ if the first two constraints are satisfied. From equation (47) we therefore require that the capital gains on the industry portfolio are equal to the intertemporal and labour terms of trade effect. We summarise this argument in the next Lemma.

Lemma 9 *An industry portfolio* $\mathbf{s}_k(t)$ *is closed with respect to country* k *if it satisfies the following linear relation*

$$\begin{bmatrix} \mathbf{l}(t) \\ \mathbf{p}(t)\mathbf{K}(t) \\ -\dot{\mathbf{p}}(t)\mathbf{K}(t) \end{bmatrix}\mathbf{s}_k(t) = \begin{bmatrix} l_k \\ h_k(t) \\ \left(\begin{array}{c} \int_t^\infty \dot{r}(\tau)h_k(\tau)e^{-\int_t^\tau r(v)dv}d\tau + \int_t^\infty \dot{w}(\tau)l_k e^{-\int_t^\tau r(v)dv}d\tau \\ -\int_t^\infty \dot{\mathbf{p}}(\tau)\mathbf{c}_k(\tau)e^{-\int_t^\tau r(v)dv}d\tau \end{array}\right) \end{bmatrix} \tag{48}$$

Then, given any closed industry portfolio $\mathbf{s}_k(t)$*, country* k *'s Real Income can be calculated as*

$$RI_k(t) = r_k^R(t)h_k(t) + w_k^R(t)l_k - \dot{\mathbf{p}}_k^R(t)\mathbf{K}(t)\mathbf{s}_k(t). \tag{49}$$

Proof Follows immediately by substituting out for the terms of trade in equation (36) in terms of the capital gains on the closed industry portfolio. ■

[16] This is single linear constraint. It is a therefore a far weaker constraint than, say, requiring $\mathbf{x}_k(t)$ to be proportional to $\mathbf{c}_k(t)$

We need to make the following remarks. In equation (48), all terms on the right hand side are a function only of the variables that are directly concerned with the country k's consumption planning problem. Thus these variables are a function of prices, country k's wealth, labour supply and consumption. In contrast, the variables on the left hand side are a function of variables associated with the firms production decisions. Therefore the closed industry portfolio $\mathbf{s}_k(t)$ can be seen as relating country k's consumption path to the production sector, or equivalently its demand for resources to the supply side.

Given a closed industry portfolio, $\mathbf{s}_k(t)$, we are able to calculate country k's Real Income as its current receipts, returns to wealth plus labour income, minus capital gains on a closed industry portfolio. However to calculate $\mathbf{s}_k(t)$ from equation (48), one needs to know the future equilibrium path of the economy. In practice, it will therefore be necessary to make some approximation in order to estimate $\mathbf{s}_k(t)$. The most obvious is to assume that industry portfolio is relatively stable over time. Or more precisely, that changes in the closed industry portfolio are small relative to changes in the capital stock. This seems a reasonable assumption if the all countries' incomes are growing at roughly the same rate. Armed with this assumption, multiply equation (46) on the left by $\dot{\mathbf{p}}(t)$ and rewrite as

$$\dot{\mathbf{p}}(t)\mathbf{Y}(t)\mathbf{s}_k(t) = \dot{\mathbf{p}}(t)\left(\mathbf{c}_k(t) + \dot{\mathbf{K}}(\mathbf{t})\mathbf{s}_k(t) + \mathbf{K}(\mathbf{t})\dot{\mathbf{s}}_k(t) + \mathbf{x}_k(t)\right) \tag{50}$$

which is a first order differential equation in $\mathbf{s}_k$; the solution to which is the terms of trade balance (47). Now if we assume the closed industry portfolio evolves slowly over time, we can assume the third term on the right hand side is small relative to the others and so

$$\dot{\mathbf{p}}(t)\left(\mathbf{Y}(\mathbf{t}) - \dot{\mathbf{K}}(\mathbf{t})\right)\mathbf{s}_k(t) \approx \dot{\mathbf{p}}(t)\mathbf{c}_k(t) + \dot{\mathbf{p}}(t)\mathbf{x}_k(t) \tag{51}$$

To ensure there is no external terms of trade effect, we need to find an industry portfolio such that $\dot{\mathbf{p}}(t)\mathbf{x}_k(t) = 0$ for all t. However from equation (51), a reasonable approximation to such a portfolio is one whose output prices, net of any investment, change at the same rate as the country's consumption bundle. Therefore an industry portfolio satisfying the following equation

$$\begin{bmatrix} \mathbf{l}(t) \\ \mathbf{p}(t)\mathbf{K}(t) \\ \dot{\mathbf{p}}(t)\left(\mathbf{Y}(\mathbf{t}) - \dot{\mathbf{K}}(\mathbf{t})\right) \end{bmatrix} \mathbf{s}_k(t) = \begin{bmatrix} l_k \\ h_k(t) \\ \dot{\mathbf{p}}(t)\mathbf{c}_k(t) \end{bmatrix} \tag{52}$$

is likely to be a good approximation to a closed industry portfolio. This equation, in contrast to equation (48), is entirely in terms of currently observed variables[17]. Therefore one feasible approach to estimating a country's Real Income is to first estimate a closed industry portfolio from equation (52). In this case, Real Income can be estimated directly from equation (49) as total receipts minus the capital gains on this closed industry portfolio.

[17]It might appear from this equation that the closed industry portfolio, $\mathbf{s}_k$, depends on the price normalisation. However if the closed industry portfolio satisfies these equations for one price normisation, it will satisfy them for any other price normalisation (subject to our earlier approximation). See the proof of Lemma 9 in the Appendix .

We have not proved the existence of a closed industry portfolio for country k. Yet equation (48), or the approximation (52), is linear in $\mathbf{s}_k(t)$ and so the existence of 3 firms with sufficiently different technologies should be sufficient to imply the existence of a closed industry portfolio $\mathbf{s}_k(t)$. If there are more than 3 firms, the portfolio will not be unique; so one could impose additional constraints. The most obvious is a no-short constraint, i.e. the holdings are non-negative, $s_{ik} \geq 0$ for all i. Then among this set of portfolios the one that is most likely to be of interest is the portfolio closest to a given domestic portfolio. If we assume, as before, that firm i is associated with country k, then Real Income would be equal to output of firm i, or GDP, plus an adjustment. This adjustment would be equal to the net property income minus the capital gains on the net foreign industry portfolio, that is $r(\mathbf{pKs}_k - \mathbf{pk}_i) - \dot{\mathbf{p}}(\mathbf{Ks}_k - \mathbf{pk}_i)$.

One final question is whether there exists of a set of closed industry portfolios, one for each country, that allocate all the economy's capital to one or other of the country. Further if we demand that all these portfolios be non-negative, $s_{ik} \geq 0$ for all i and k, this is equivalent to the existence of stochastic matrix satisfying a set of linear equations. Using the same convention as above, define the matrices $\mathbf{S} = [\mathbf{s}_1\, \mathbf{s}_2 \cdots \mathbf{s}_k \cdots]$, $\mathbf{C} = [\mathbf{c}_1\, \mathbf{c}_2 \cdots \mathbf{c}_k \cdots]$ and the vector $\mathbf{h} = [h_1\ h_2 \cdots\ h_k \cdots]$. The existence of a set of closed industry portfolios, implies the matrix $\mathbf{S}$ must satisfies the equation

$$\begin{bmatrix} \mathbf{l}(t) \\ \mathbf{p}(t)\mathbf{K}(t) \\ \dot{\mathbf{p}}(t)\left(\mathbf{Y}(\mathbf{t}) - \dot{\mathbf{K}}(\mathbf{t})\right) \end{bmatrix} \mathbf{S}(t) = \begin{bmatrix} \mathbf{l}(t) \\ \mathbf{h}(t) \\ \dot{\mathbf{p}}(t)\mathbf{C}(t) \end{bmatrix} \tag{53}$$

Further if these portfolios are to allocate all the economy's capital and output, then the matrix $\mathbf{S}$ must also satisfy the constraint $\mathbf{S}\ \mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is a vector of 1s. This latter condition coupled with the no-short constraint, $s_{ik} \geq 0$, implies that $\mathbf{S}$ is a stochastic matrix. Such matrices have a number of interesting properties; see for example Horn & Johnson (1985). Hence the problem of estimating the Real Income for each country amounts to finding a stochastic matrix $\mathbf{S}$ that satisfies equation (53).

6.3 Comparison with Existing Practice

The conclusions flowing from the above are sharply at odds with existing practice. They show income being measured in consumption terms in both closed and open economies. The United Nations *System of National Accounts* (1993, pp 404-406) suggests adjusting net domestic product (a volume measure of output net of depreciation) by deducting exports deflated by the price index of exports less imports deflated by the price index of imports and adds back the external trade balance deflated by a single deflator. While it discusses the possibility of using the consumption deflator for this purpose it also states 'an average of the import and export price indices is likely to provide a suitable deflator'. Net factor and transfer incomes received from abroad are, however, supposed to be deflated by the price index of total final expenditure and there is also a suggestion that this deflator might be used for the trade balance.

Thus even in a closed economy the measure of net income differs from what we describe here because output of net investment goods is deflated by a price index for investment goods rather than the consumption price index. When discussing income, investment should be measured in terms of consumption foregone. In an open economy the difference is accentuated because the *System of National Accounts* does not advocate use of the consumption deflator for factor and transfer income from abroad, and mentions it as only one of a number of possibilities for deflating the trade balance.

When dealing with exhaustible resources the *System of Environmental and Economic Accounts* ignores the terms of trade effect which benefits a resource exporter and thus leads to the erroneous conclusion that the net income of a pure resource exporter is zero. As a counterpart it ignores the adverse terms of trade effect faced by resource-importing countries and thus overstates their incomes. Of course it may be objected that in practice the prices of exhaustible resources do not follow the Hotelling Rule but any faults arising from this are likely to matter less than the conclusion that the income of a pure resource exporter is zero.

The weakness of the UN measures can be seen from invariance tests. A capital-rich country makes an arbitrary choice between employing its capital abroad and exporting capital-intensive goods; a satisfactory measure of income should be invariant between these choices. A resource-rich country can choose between selling the resource and living on the income from investing the proceeds or gradually extracting the resource. The measure of income it records should be unaffected by this choice (Sefton and Weale, 1996). Our measures meet these invariance tests while the UN measures do not.

Of course in the open economy, our measure suffers from the need to predict future changes in the terms of trade in order to estimate current income. The prediction follows immediately from the equilibrium conditions where exhaustible resources are concerned but is less obvious otherwise. The approximation we suggest to deal with this, in equation (53) can, however, be estimated from existing OECD data on the capital stocks and trade flows of the different countries. At the very least it might be hoped that some indication could be obtained of the likely magnitude of the effect.

7 Conclusions

The main conclusion from this study is that the concept of aggregate income, far from being a purely macro-economic concept or one relying on a representative consumer, can be linked to aggregate welfare in a general equilibrium economy consisting of finitely many heterogeneous infinitely-lived agents. We show that a measure which emerges naturally from an examination of the production side of the general equilibrium is also consistent with a definition that income is level of consumption at which the presented discounted sum of current and future consumption is unchanging- a definition consistent with Hicks' idea that income is the level of consumption possible which leave the consumer as well off at the end of the week as he had been at the start.

Our measure of income, which we call Real Income resolves a number of previously unsatisfactorily unanswered questions. First we show how the concept of income relates to current and future consumption explaining the difference between this and measures of income based on sustainability; secondly we show that there is a simple additive link between the incomes of households relate to a measure of national income or social income; and finally we show how the concept of national income relates to national product in both closed and open economies.

Our framework also allows us to link capital gains to expected future changes in factor prices and show, at least in the economy as we model, it that capital gains should not be added on to income as conventionally measured. Our income measure meets two important invariance conditions which conventional national accounting measures do not fulfil. First real income is unaffected by whether a country with an excess of one factor of production addresses this by exporting the factor or by exporting goods produced using the factor intensively. Secondly the real income of a resource owner is the same whether the owner extracts the resource gradually or sells the extraction rights and draws income from investing the proceeds.

This article immediately suggests some extensions. The first is relatively trivial, and is to extend the framework to allow for exogenous productivity growth along the lines suggested by Weitzman (1997) in his analysis of income as sustainable consumption. Real Income could be defined in terms of a weighted average of growth adjusted current and future consumption. All the propositions will then need modifying. The other two extensions are more involved. Mirrlees (1969) was the first to examine the problem of defining National Income in an economy with imperfect competition. This idea could be carried over to the general equilibrium framework. Our analysis of the household sector would, in all likelihood, carry over with only minor modifications. However the analysis of the production sector, far from being a trivial application of the assumption of Constant Returns to Scale production, would be become far more involved. Another extension, though similar in some respects to the second, is to include a government sector. This introduces two complications; first market prices, proportional to marginal utility, are no longer equal to factor prices. Thus the analysis of the production side holds true when outputs are value at the prices received for them by producers, factors are charged at the prices paid by producers and producers do not have market power. But in the presence of indirect taxes or private sector monopoly power this will not be the case. Secondly there is the issue of how households value both collective and non-collective government consumption in utility terms; only if public consumption is valued on an equal basis with private consumption in the social welfare function can the model set out here be easily applied. Both these issues need to be addressed in order to develop a fully comprehensive framework for the concept of income.

A Appendix

Discussion of the existence of an equilibrium for the economy. This discussion is based heavily on Kehoe et al. (1990). Their discrete-time economy consists of a finite number of infinitely-lived heterogeneous households and a single production technology. Households act so as to maximise their welfare, the firm to maximise their profits over an infinite horizon. Under a standard set of assumptions on preferences and technology, they prove that the equilibria are locally unique for almost all endowments.

Our model differs in a number of respects; first we have set our problem in continuous time, and second our model has a finite number of firms and goods. However we do not require the equilibrium to be determinate for our results to be valid, merely that an equilibrium exists which satisfies the first-order conditions. In fact, a requirement of any definition of Income should be that its is invariant to most sources of indeterminacy. To illustrate, assume that two firms have identical technology. Hence there will be an infinite number of equilibria which differ in the ratio of the production of firm 1 to firm 2. Yet the income of any household should not depend on the particular location of the production.

Now in order to prove existence of an equilibrium, it is only necessary that the associated value function for the social welfare problem to be continuously differentiable. But as Benveniste & Scheinkman (1979) prove for both the discrete and continuous time cases, the value function is differentiable under a set of relatively weak conditions. These require the utility functions to be concave and differentiable, and that the solution path be an interior solution to the maximisation problem. These conditions are therefore sufficient for the Negishi approach to proving the existence of a social welfare function and an equilibrium to be valid.

The following are a set of assumptions modified from Kehoe et al. (1990) to our framework. We include them as a guide only. Proving that these assumptions are sufficient to guarantee the existence of an equilibrium is beyond the scope of this paper.

Assumption 1.

(i) The utility function is concave, strictly increasing, continuous, and smooth on the positive orthant.

(ii) The marginal utility of consumption tends to infinity as consumption tends to zero, $\lim_{\lambda\to 0} \frac{\partial u_k(\lambda \mathbf{c}_k)}{\partial \lambda} = \infty$

Assumption 2.

(i) The production functions exhibit constant returns to scale, i.e. $g_i(\lambda \mathbf{k}_i, \lambda l_i, \lambda \mathbf{y}_i) = \lambda g_i(\mathbf{k}_i, l_i, \mathbf{y}_i)$

(ii) The production functions are continuous and strictly concave, i.e.

$$g_i(\mathbf{k}_i, l_i, \mathbf{y}_i) \geq \mu g_i\left(\hat{\mathbf{k}}_i, \hat{l}_i, \hat{\mathbf{y}}_i\right) + (1-\mu) g_i\left(\hat{\mathbf{k}}_i, \hat{l}_i, \hat{\mathbf{y}}_i\right)$$

for all $\mu \in [0,1]$ and strictly greater than for some $\mu \in [0,1]$ where $(\mathbf{k}_i, l_i, \mathbf{y}_i) = \mu\left(\hat{\mathbf{k}}_i, \hat{l}_i, \hat{\mathbf{y}}_i\right) + (1-\mu)\left(\hat{\mathbf{k}}_i, \hat{l}_i, \hat{\mathbf{y}}_i\right)$, and are smooth on the positive orthant.

(iii) The production functions exhibit non-negative marginal returns to all input factors and a non-negative marginal costs to all outputs,

$$\frac{\partial g_i}{\partial k_{ij}} \geq 0 \qquad \frac{\partial g_i}{\partial l_i} \geq 0 \quad \text{and} \quad \frac{\partial g}{\partial y_{ij}} \leq 0 \tag{54}$$

(iv) The production functions satisfy the Inada growth conditions,

$$g_i(0, l_i, \mathbf{y}_i) = 0 \qquad \lim_{\lambda \to 0} \frac{\partial g_i(\lambda \mathbf{k}_i, l_i, \mathbf{y}_i)}{\partial \lambda} = \infty \tag{55}$$

for all bounded and positive l_i,and $\mathbf{k}_i$.

The final growth conditions are to ensure that no firm seizes production along the equilibrium path.

The final set of assumptions are to ensure that the economy's equilibrium is an interior solution to the maximisation problem. To state these condition precisely it is necessary to distinguish between four types of goods; consumption goods, exhaustible, resource goods, reproducible and non-reproducible capital goods. For each of these goods we state a sufficient condition to ensure that either the supply of these goods or the stock of associated capital does not equal zero at any point along the equilibrium path; these conditions should be sufficient to ensure an interior solution. For this we need to define the n_j unit vector $\mathbf{e}_j$ as the vector whose j^{th} element is equal to 1 and all other elements are 0.

Assumption 3.

(i) A good is a consumption good if it is not used as a capital good in the production process, i.e. for every firm $g_i\left(\mathbf{k}_i+\lambda\mathbf{e}_j, l_i, \mathbf{y}_i\right)=g_i\left(\mathbf{k}_i, l_i, \mathbf{y}_i\right)$ for all $l_i, \mathbf{y}_i$ and $\mathbf{k}_i$, and the initial stock of the good is zero, i.e. $\mathbf{e}_j\mathbf{k}(t)=0$. In this case we shall assume that the consumption good is an essential consumption good for at least one household, in that there exists a household k such that

$$\lim_{c_{kj}\to 0}\frac{\partial u_k(\mathbf{c}_k)}{\partial c_{kj}}=\infty$$

on the positive orthant.

(ii) A good is an exhaustible resource good if it is not used as a capital good in the production process, i.e. for every firm $g_i\left(\mathbf{k}_i+\lambda\mathbf{e}_j, l_i, \mathbf{y}_i\right)=g_i\left(\mathbf{k}_i, l_i, \mathbf{y}_i\right)$ for all $l_i, \mathbf{y}_i$ and $\mathbf{k}_i$, there is an initial stock of the good, i.e. $\mathbf{e}_j\mathbf{k}(t)>0$, and it can not be manufactured, $\mathbf{e}_j\mathbf{y}_i(t)\leq 0$ for all i. In this case we shall assume that the good is an essential resource good for at least one firm, in that is there exists a firm i such that

$$\lim_{y_{ij}\to 0}\frac{\partial g_i\left(\mathbf{k}_i, l_i, \mathbf{y}_i\right)}{\partial y_{ij}}=\infty$$

for all bounded feasible $\mathbf{y}_i$ and bounded and positive l_i,and $\mathbf{k}_i$.

(iii) A good is a non-reproducible capital good if it is used in the production process by at least one firm, i.e. $\frac{\partial g_i}{\partial k_{ij}}>0$ for all bounded and positive l_i,and $\mathbf{k}_i$, there is an initial stock of the good, i.e. $\mathbf{e}_j\mathbf{k}(t)>0$, and it is non-exhaustible, $\mathbf{e}_j\mathbf{y}_i(t)=0$ for all i.

(iv) A good is a reproducible capital good if it is used in the production process by at least one firm, i.e. $\frac{\partial g_i}{\partial k_{ij}}>0$ for all bounded and positive l_i,and $\mathbf{k}_i$, there is an initial stock of the good, i.e. $\mathbf{e}_j\mathbf{k}(t)>0$ and it is manufactured by at least one firm i, $\mathbf{e}_j\mathbf{y}_i(t)>0$. In this case we shall assume that the capital good is an essential capital good for at least one firm, in that is there exists a firm i such that

$$\lim_{k_{ij}\to 0}\frac{\partial g_i\left(\mathbf{k}_i, l_i, \mathbf{y}_i\right)}{\partial k_{ij}}=\infty \quad \text{or} \quad \lim_{k_{ij}\to\infty}\frac{\partial g_i}{\partial k_{ij}}/k_{ij}<1$$

for all bounded feasible $\mathbf{y}_i$ and bounded and positive l_i,and $k_{ij'}$, $j'\neq j$.

To prove determinacy of the equilibria, we need the value function to be twice continuously differentiable. Whether this is likely to be satisfied in a multi-sector discrete time growth model is discussed in detail in Section 6 of Kehoe et al. (1990). Further they suggest in the note at the end of the paper, that the additional interiority and differentiability conditions of Santos (1992) are sufficient to guarantee that the value function is twice continuously differentiable. More recently Shannon & Zames (2002) looked at the problem in a more general setting. They introduce a condition called quadratic concavity, which requires that near any feasible bundle, utility differs from the linear approximation by an amount that is at least quadratic in the distance to the given bundle. This condition, in conjunction with the normal concavity conditions on the utility functions, is a sufficient to rule out the problem of perfect substitutes, and prove determinacy in a more general framework.

Proof of Proposition 1. To derive this relationship, we begin by differentiating the production function (equation 1) for firm i with respect to time,

$$\frac{dg_i}{dt} = \frac{\partial g}{\partial \mathbf{y}_i}\dot{\mathbf{y}}_i + \frac{\partial g}{\partial \mathbf{k}_i}\dot{\mathbf{k}}_i + \frac{\partial g}{\partial l_i}\dot{l}_i = 0 \tag{56}$$

Now a requirement for an equilibrium is that marginal products are proportional to their respective prices. Hence equation (56) implies all firms in the economy, with $y = \sum_{i=1}^{n} y_i$, $\dot{k} = \sum_{i=1}^{n} \dot{k}_i$ and, with a fixed overall labour force, $\sum_{i=1}^{n} \dot{l}_i = 0$ to show that

$$\mathbf{p}\dot{\mathbf{y}} = \boldsymbol{\rho}\dot{\mathbf{k}} = \boldsymbol{\rho}(\mathbf{y} - \mathbf{c})$$

If we divide through by the Divisia price index Π, to put everything in real prices and then substitute out for rental prices using the arbitrage condition (2) we can derive the following equation that is satisfied along the equilibrium path

$$\mathbf{p}^R\dot{\mathbf{y}} = (r^R\mathbf{p}^R - \dot{\mathbf{p}}^R)(\mathbf{y} - \mathbf{c}).$$

By definition $\dot{p}^R c = 0$, and therefore rearranging implies

$$\frac{d}{d\tau}\left(\mathbf{p}^R(\tau)\mathbf{y}(\tau)e^{-\int_t^\tau r^R(v)dv}\right) = -r^R(\tau)\mathbf{p}^R(\tau)\mathbf{c}(\tau)e^{-\int_t^\tau r^R(v)dv}$$

Integrating both sides from $\tau = t$ to ∞ along the equilibrium path and using the transversality condition for an equilibrium yields the result. ■

Proof of Proposition 3. We first note the differential

$$\frac{d}{dt}\left(\int_t^\infty u_k(\mathbf{c}_k)\, e^{-\theta(\tau - t)}d\tau\right) = \int_t^\infty \frac{d}{d\tau}\left(u_k(\mathbf{c}_k)\right) e^{-\theta(\tau-t)}d\tau \tag{57}$$

as the discount factor is constant. We replace the utility function by its indirect equivalent on the right side, and differentiate to show,

$$\frac{d}{dt}\left(\int_t^\infty u_k(\mathbf{c}_\mathbf{k})\, e^{-\theta(\tau-t)}d\tau\right) = \int_t^\infty \frac{d}{d\tau} z_k(m_k^R, \mathbf{p}^\mathbf{R})\mathbf{e}^{-\boldsymbol{\theta}(\boldsymbol{\tau}-\mathbf{t})}\mathbf{d}\boldsymbol{\tau} \tag{58}$$

$$= \int_t^\infty \left(\frac{\partial z_k}{\partial m_k^R}\dot{m}_k^R + \frac{\partial z_k}{\partial \mathbf{p}^R}'\dot{\mathbf{p}}^R\right) e^{-\theta(\tau-t)}d\tau. \tag{59}$$

We can substitute out using Roy's identity in equation (13) and then simplify the expression further by noting that $\dot{p}^{R\prime} c_k = 0$. Hence

$$\frac{d}{dt}\left(\int_t^\infty u_k(\mathbf{c}_k)\, e^{-\theta(\tau-t)} d\tau\right) = \int_t^\infty \frac{\partial z_k}{\partial m_k^R} \dot{m}_k^R e^{-\theta(\tau-t)} d\tau$$

Integrating the right hand side by parts gives

$$\frac{d}{dt}\left(\int_t^\infty u_k(\mathbf{c}_k)\, e^{-\theta(\tau-t)} d\tau\right) = \left[\frac{\partial z_k}{\partial m_k^R} m_k^R e^{-\theta(\tau-t)}\right]_t^\infty - \int_t^\infty m_k^R \frac{d}{d\tau}\left(\frac{\partial z_k}{\partial m_k^R} e^{-\theta(\tau-t)}\right) d\tau \quad (60)$$

Before proceeding any further, we must first integrate the intertemporal efficiency condition (expressed in real prices) between t and $\tau \geq t$ in equation (15) to derive the identity

$$\frac{\partial z_k}{\partial m_k^R}(\tau) e^{-\theta(\tau-t)} = \frac{\partial z_k}{\partial m_k^R}(t) e^{-\int_t^\tau r^R(v)dv}$$

We now differentiate this identity with respect to τ to show

$$\frac{d}{d\tau}\left(\frac{\partial z_k}{\partial m_k^R}(\tau) e^{-\theta(\tau-t)}\right) = -r(\tau) \frac{\partial z_k}{\partial m_k^R}(t) e^{-\int_t^\tau r^R(v)dv}$$

Substituting this expression for the derivative into equation (60), and writing $m_k^R(t)$ as $p^R(t)' c_k(t)$ gives the final result. ■

Proof of Proposition 4. These aggregation results are relatively trivial. Aggregate real consumption is by definition the sum of the reflated real, or nominal, household consumption deflated by the aggregate deflator. Similarly from the definition for the household deflators

$$\frac{\dot{\Pi}_k(t)}{\Pi_k(t)} = \frac{\dot{\mathbf{p}}(t)\mathbf{c}_k(t)}{\mathbf{p}(t)\mathbf{c}_k(t)}$$

it is immediately apparent that

$$\sum_k \frac{\mathbf{p}(t)\mathbf{c}_k(t)}{\mathbf{p}(t)\mathbf{c}(t)} \frac{\dot{\Pi}_k(t)}{\Pi_k(t)} = \frac{\dot{\mathbf{p}}(t)\mathbf{c}(t)}{\mathbf{p}(t)\mathbf{c}(t)} = \frac{\dot{\Pi}(t)}{\Pi(t)}.$$

Therefore from equation (5), the aggregate real interest rate is the sum of the consumption weighted household real interest rates. The aggregation result for real income requires a little more work. Integrating the equation define the real interest rate, equation (5), implies that the identity

$$\frac{\Pi_k(\tau)}{\Pi_k(t)} e^{-\int_t^\tau r(v)dv} = e^{-\int_t^\tau r_k^R(v)dv} \quad (61)$$

is satisfied for each household interest rate and the aggregate interest rate too. Therefore we can write the sum of the reflated household real incomes as

$$\begin{aligned}
\sum_k \frac{\Pi_k(t)}{\Pi(t)} RI_k^R(t) &= \sum_k \frac{\Pi_k(t)}{\Pi(t)} \int_t^\infty r_k^R(\tau)\mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau \\
&= \sum_k \frac{1}{\Pi(t)} \int_t^\infty r_k^R(\tau)\mathbf{p}(\tau)\mathbf{c}_k(\tau) e^{-\int_t^\tau r(v)dv} d\tau.
\end{aligned}$$

It is therefore sufficient to demonstrate that $\sum_k r_k^R(\tau)p(\tau)c_k(\tau) = r^R(\tau)p(\tau)c(\tau)$ to complete the result. However this is just a restatement of the real interest rate aggregation result. Therefore the income aggregation result follows after one further substitution of identity (61) relating the nominal interest rate back to the aggregate real interest rate. ■

Proof of Proposition 5. From the definition of the social welfare function and Proposition 3, we have

$$\begin{aligned}
\frac{d}{dt}\left(\int_t^\infty u(\mathbf{c})\, e^{-\theta(\tau-t)}d\tau\right) &= \sum_{k=1}^m \alpha_k \frac{d}{dt}\left(\int_t^\infty u_k(\mathbf{c}_k)\, e^{-\theta(\tau-t)}d\tau\right) \\
&= \sum_{k=1}^m \alpha_k z_k^{R\prime}(t)\left(RI_k^R(t) - \mathbf{p}_k^R(t)\mathbf{c}_k(t)\right)
\end{aligned}$$

We can now use the properties of the marginal utility money, stated in equations (33) and (14), to show

$$\begin{aligned}
\frac{d}{dt}\left(\int_t^\infty u(\mathbf{c})\, e^{-\theta(\tau-t)}d\tau\right) &= \sum_{k=1}^m \alpha_k z_k'(t)\Pi_k(t)\left(RI_k^R(t) - \mathbf{p}_k^R(t)\mathbf{c}_k(t)\right) \\
&= z'(t)\sum_{k=1}^m \Pi_k(t)\left(RI_k^R(t) - \mathbf{p}_k^R(t)\mathbf{c}_k(t)\right) \\
&= z'(t)\Pi(t)\sum_{k=1}^m \frac{\Pi_k(t)}{\Pi(t)}\left(RI_k^R(t) - \mathbf{p}_k^R(t)\mathbf{c}_k(t)\right).
\end{aligned}$$

The result then follows by substituting in the real consumption and income aggregation results proved in Proposition 4, and rewriting the marginal utility of money in real terms using equation (14). ■

Proof of equation (23). Here we prove the relationship given in equation (23) between Real income and income as the stationary equivalent of future consumption, $AI(t)$. To prove this we first derive a couple of identities that relate the long run interest rate to the nominal interest rate. The first of these follows directly by differentiating the equation defining the long run interest rate, equation (21), to give

$$\frac{\dot{r_L}(t)}{r_L(t)} = r_L(t) - r(t). \tag{62}$$

This equation can be integrated over the interval t to τ to give

$$r_L(t)e^{-\int_t^\tau r(v)dv} = r_L(\tau)e^{-\int_t^\tau r_L(v)dv} \tag{63}$$

We can now substitute this identity into the definition of income as the stationary equivalent of future consumption, $AI(t)$, proving that

$$\begin{aligned}
AI(t) &= \int_t^\infty r_L(t)\mathbf{c}(\tau)e^{-\int_t^\tau r(v)dv}d\tau. \\
&= \int_t^\infty r_L(\tau)\mathbf{c}(\tau)e^{-\int_t^\tau r_L(v)dv}d\tau.
\end{aligned} \tag{64}$$

which is the claim made in the footnote on page 10, that $AI(t)$ is a weighted average of current and future consumption. We can now integrate the second line by parts

$$\begin{aligned} AI(t) &= \mathbf{c}(t) + \int_t^\infty \dot{\mathbf{c}}(\tau) e^{-\int_t^\tau r_L(v)dv} d\tau & (65) \\ &= \mathbf{c}(t) + \int_t^\infty \frac{r_L(t)}{r_L(\tau)} \dot{\mathbf{c}}(\tau) e^{-\int_t^\tau r(v)dv} d\tau & (66) \end{aligned}$$

where the second line comes from substituting back from equation (63). Equation (23) now follows directly from this identity and the third identity in Lemma 7. ■

Proof of Lemma 7. Equation (35) describes the relationship between Real Income and 'income as the return to wealth'. We first derive this identity. Consumption wealth is the solution to the differential equation

$$\frac{d}{d\tau} CW_k^R(\tau) = r_k^R(\tau) CW_k^R(\tau) - \mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau)$$

integrated over the interval $[t, \infty)$. This differential equation can be rewritten as

$$\frac{d}{d\tau}\left(CW_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv}\right) = -\mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau) e^{-\int_t^\tau r_k^R(v)dv}.$$

Now to derive the identity, substitute out for the present value of consumption in the definition of Real Income using the above equation, to show that

$$RI_k^R(t) = -\int_t^\infty r_k^R(\tau) \frac{d}{d\tau}\left(CW_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv}\right) d\tau. \tag{67}$$

This integral can be integrated by parts, so that

$$RI_k^R(t) = \left[-r_k^R(\tau) CW_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv}\right]_t^\infty + \int_t^\infty \dot{r}_k^R(\tau) CW_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv} d\tau. \tag{68}$$

giving the result.

The second identity is derived from the first by substituting out for consumption wealth in terms of financial plus human capital wealth. If we denote the human capital wealth as $hc_k^R(t) = \int_t^\infty w_k^R(\tau) l_k e^{-\int_t^\tau r_k^R(v)dv} d\tau$ then its differential can be written

$$\frac{d}{d\tau}\left(hc_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv}\right) = -w_k^R(\tau) l_k e^{-\int_t^\tau r_k^R(v)dv}.$$

We can now substitute out for consumption wealth in equation (67) to derive the following expression for Real Income of household k

$$\begin{aligned} RI_k^R(t) &= -\int_t^\infty r_k^R(\tau) \frac{d}{d\tau}\left(h_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv}\right) d\tau + \int_t^\infty r_k^R(\tau) w_k^R(\tau) l_k e^{-\int_t^\tau r_k^R(v)dv} d\tau \\ &= -\int_t^\infty r_k^R(\tau) \frac{d}{d\tau}\left(h_k^R(\tau) e^{-\int_t^\tau r_k^R(v)dv}\right) d\tau - \int_t^\infty w_k^R(\tau) l_k \frac{d}{d\tau}\left(e^{-\int_t^\tau r_k^R(v)dv}\right) d\tau \end{aligned}$$

The first term on the right hand side is integrated by parts in a manner identical to equation (68). The second term is also integrated by parts but this time the integral is partitioned differently. This partition is clear form the equation above. These integrations give

$$\begin{aligned} RI_k^R(t) &= \left[-r_k^R(\tau)h_k^R(\tau)e^{-\int_t^\tau r_k^R(v)dv}\right]_t^\infty + \int_t^\infty \dot{r}_k^R(\tau)h_k^R(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau + \\ &\left[-w_k^R(\tau)l_k e^{-\int_t^\tau r_k^R(v)dv}\right]_t^\infty + \int_t^\infty \dot{w}_k^R(\tau)l_k e^{-\int_t^\tau r_k^R(v)dv}d\tau \end{aligned}$$

and the result follows immediately.

The final identity in the Lemma is derived in a very similar way. Split up the integral in the definition of real income as

$$RI_k^R(t) = \int_t^\infty r_k^R(\tau)\mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau)e^{-\int_t^\tau r_k^R(v)dv}d\tau = -\int_t^\infty \mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau)\frac{d}{d\tau}\left(e^{-\int_t^\tau r_k^R(v)dv}\right)d\tau$$

we can integrate by parts for the final time to derive the expression

$$RI_k^R(t) = \left[-\mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau)e^{-\int_t^\tau r_k^R(v)dv}\right]_t^\infty + \int_t^\infty \frac{d}{d\tau}\left(\mathbf{p}_k^R(\tau)\mathbf{c}_k(\tau)\right)e^{-\int_t^\tau r_k^R(v)dv}d\tau.$$

The final result follows after noting the real prices are defined so that $\dot{\mathbf{p}}_k^R(t)c_k(t) = 0$. ∎

Proof of Lemma 8 We start by noting that as each firm is assumed to exhibit CRS, output and factor prices satisfy the identity

$$\dot{\mathbf{p}}\mathbf{y}_i = \dot{\boldsymbol{\rho}}\mathbf{k}_i + \dot{w}l_i \tag{69}$$

for each firm. Therefore, a similar identity is satisfied

$$\dot{\mathbf{p}}\mathbf{y}_k = \dot{\boldsymbol{\rho}}\mathbf{k}_k + \dot{w}l_k \tag{70}$$

for any industry portfolio too. Substituting in the goods balance, equation (46), into this equation gives

$$\dot{\mathbf{p}}\left(\mathbf{c}_k + \dot{\mathbf{k}}_k + \mathbf{x}_k\right) = \dot{\boldsymbol{\rho}}\mathbf{k}_k + \dot{w}l_{k.} \tag{71}$$

Now we can substitute out the change in rental prices using the intertemporal efficiency condition (2), and rearrange to derive the equation

$$\dot{\mathbf{p}}\mathbf{c}_k + \dot{\mathbf{p}}\mathbf{x}_k = \dot{r}\mathbf{p}\mathbf{k}_k + \dot{w}l_{k.} + \left(r\dot{\mathbf{p}}\mathbf{k}_k - \ddot{\mathbf{p}}\mathbf{k}_k - \dot{\mathbf{p}}\dot{\mathbf{k}}_k\right) \tag{72}$$

Multiplying this equation through by the discount factor $e^{-\int_t^\tau r_k^R(v)dv}$ and noting the bracketed terms on the right hand side is a perfect differential, we can derive the relation

$$\left(\dot{\mathbf{p}}\mathbf{c}_k + \dot{\mathbf{p}}\mathbf{x}_k\right)e^{-\int_t^\tau r_k^R(v)dv} = \left(\dot{r}\mathbf{p}\mathbf{k}_k + \dot{w}l_{k.}\right)e^{-\int_t^\tau r_k^R(v)dv} - \frac{d}{d\tau}\left(\dot{\mathbf{p}}\mathbf{k}_k e^{-\int_t^\tau r_k^R(v)dv}\right). \tag{73}$$

The result follows by integrating this expression from $\tau = t$ to ∞ and using the transversality condition. ■

Proof of Lemma 9. Follows immediately by substituting out for the terms of trade in equation (36) in terms of the capital gains on the closed industry portfolio. The results is then expressed in real prices. Of interest is the implication of this result, that if $\dot{\mathbf{p}}\mathbf{x}_k = 0$ for one price base then it equals zero for any price base. To understand this result, write the budget identity for country k as

$$\dot{h}_k(t) = r(t)h_k(t) + w(t)l_k - \mathbf{p}(t)\mathbf{c}_k(t).$$

Now if the first two constraints in equation (48) are satisfied, we can write

$$\dot{h}_k(t) = \dot{\mathbf{p}}(t)\mathbf{k}_k(t), \;\; l_k = \widetilde{l}_k \qquad \text{and} \qquad \dot{h}_k(t) = \dot{\mathbf{p}}(t)\mathbf{k}_k(t) + \mathbf{p}(t)\dot{\mathbf{k}}_k(t).$$

These relations can be substituted into the budget equation and rearranged to give

$$\mathbf{p}(t)\dot{\mathbf{k}}_k(t) + \mathbf{p}(t)\mathbf{c}_k(t) = r(t)\mathbf{p}(t)\mathbf{k}_k(t) - \dot{\mathbf{p}}(t)\mathbf{k}_k(t) + w(t)\widetilde{l}_k.$$

The first two terms of the right hand side are equal to the rents on the the capital stock $\mathbf{k}_k(t)$. Therefore the right hand side is equal to the income of industry portfolio k, which equals the value of its net output , $\mathbf{p}(t)\mathbf{y}_k(t)$. Therefore given the first two constraints in equation (48) are satisfied, we have the identity that

$$\mathbf{p}(t)\left(\dot{\mathbf{k}}_k(t) + \mathbf{c}_k(t)\right) = \mathbf{p}(t)\mathbf{y}_k(t)$$

Now multiplying the goods balance, equation (46), by the rate of change in prices gives

$$\dot{\mathbf{p}}(t)\left(\dot{\mathbf{k}}_k(t) + \mathbf{c}_k(t) + \mathbf{x}_k(t)\right) = \dot{\mathbf{p}}(t)\mathbf{y}_k(t).$$

These two identities immediately imply that if $\dot{\mathbf{p}}(t)\mathbf{x}_k(t) = 0$ for one price base, it is satisfied for any other price base. Therefore requiring $\dot{\mathbf{p}}_k^R(t)\mathbf{x}_k(t) = 0$ to remove the external terms of trade effect is equivalent to requiring $\mathbf{p}_k^R(t)\mathbf{x}_k(t) = 0$ in conjunction with the other two constraints in equation (48). ■

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