A Theory of Momentum in Sequential Voting*

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Abstract

This paper develops a rational theory of momentum in elections with sequential voting. We analyze a two-candidate election in which some voters are uncertain about the realization of a state variable that can affect their preferences between the candidates. Voters receive private signals about the state and vote in an exogenously fixed sequence, observing the history of votes at each point. We show that there is a strict equilibrium with *Posterior-Based Voting*: each voter votes for the candidate she believes to be better at the time of casting her vote, taking into account the information revealed in prior votes. In this equilibrium, herding can occur on a candidate with positive probability, and occurs with probability approaching one in large voting games. Our results help understand and have implications for sequential voting mechanisms such as presidential primaries and roll-call voting.

Keywords: sequential voting, sincere voting, information aggregation, bandwagons, momentum, information cascades, herd behavior

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> "...when New Yorkers go to vote next Tuesday, they cannot help but be influenced by Kerry's victories in Wisconsin last week. Surely those Wisconsinites knew something, and if so many of them voted for Kerry, then he must be a decent candidate."
>
> — Duncan Watts in *Slate* Magazine

1 Introduction

Many elections take place over time. The most prominent example lies at the heart of the American presidential selection process: the primaries. A series of elections by which a party selects its candidate for the general presidential election, the primaries are held sequentially across states over a few months. On a smaller scale, but also explicitly sequential, are the roll-call voting mechanisms used by city councils and Congressional bodies. A more subtle example is the general U.S. presidential election itself, where the early closing of polls in some states introduces a temporal element into voting.

In contrast, most theoretical models of voting are static. The distinction between simultaneous and sequential elections is not just of theoretical interest, but also relevant to policy. It is often suggested that in sequential elections, voters condition their choices on the acts of prior voters. Such history dependence is believed to result in *momentum* effects: the very fact that a particular alternative is leading in initial voting rounds may induce some later voters to select it who would have otherwise voted differently. Moreover, voting behavior in primaries suggests that candidates are judged by how they perform relative to expectations: a surprisingly good performance in an early primary may generate more momentum than an anticipated victory (Popkin, 1991).

The beliefs in momentum and performance relative to expectations have shaped electoral policy and strategy. Certain U.S. states aim to hold their primaries early in the process,[1] campaign funds and media attention are disproportionately devoted to initial primaries,[2] and candidates strategically attempt to gain surprise victories. For example, Hamilton Jordan, who would become Jimmy Carter's White House Chief of Staff, outlined the importance of a surprise victory in New Hampshire in a memorandum two years before Carter's campaign: "...a strong surprise in New Hampshire should be our goal, which would have a tremendous impact on successive primaries..." This assessment is supported by the analysis of Bartels (1988), who simulates Carter's national popularity prior to his victory in New Hampshire and predicts that Carter would have lost the 1976 nomination to George Wallace had the primaries been held

[1] For instance, since 1977, New Hampshire law has stated that its primary is to be the first in the nation. As a result, the state has had to move its primary, originally in March, earlier in the year to remain the first. New Hampshire's primary was held on February 20 in 1996, February 1 in 2000, and then January 27 in 2004 to compete with front-loaded primaries in other states.

[2] Bartels (1988) and Gurian (1986) document how a substantial share of the media's attention and candidates' campaign resources are devoted to New Hampshire even though that state accounts for merely 4 out of 538 total electoral votes.

simultaneously in all states. That an unexpected amount of voter support can outweigh an anticipated victory is exemplified by the 1984 Democratic primaries featuring Gary Hart and Walter Mondale. Expected to have support in Iowa, Mondale's victory with close to 50% of the caucus votes in this state was largely overshadowed by Gary Hart's ability to garner about 17% of the votes. Even though Hart's performance in Iowa was far inferior to Mondale's on an absolute scale, Hart had performed much better than expected. This garnered him momentum, and Hart proceeded to win the following primaries in Vermont and New Hampshire (Bartels, 1988).

This paper provides a positive theory of behavior in sequential elections that accounts for the dynamics of momentum. Specifically, we consider a sequential version of a canonical election environment (Feddersen and Pesendorfer, 1996, 1997). There are two candidates, and a finite population of voters who vote in an exogenously fixed sequence, each observing the entire history of prior votes. The candidate who receives the majority of votes wins the election. There are two kinds of voters: Neutrals and Partisans. Neutrals desire to elect the "correct" or better candidate, which depends on the realization of an unknown state variable. Partisans, on the other hand, wish to elect their exogenously preferred candidate regardless of the state. Whether a voter is Neutral or Partisan is her private information, and each voter receives a private binary signal that contains some information about the unknown state.

In this environment, we formalize an informational theory of voting behavior that is built on the simple but powerful logic that if (some) initial voters use their private information in deciding how to vote, the voting history provides useful information to later voters. We show that an appealing form of history-dependent strategies constitutes fully rational behavior. This generates rich momentum effects where a leading candidate is judged relative to his expected partisan support, and some voting histories can cause future voters to entirely ignore their private information, with Neutrals simply joining "bandwagons" for one of the candidates.

At first glance, this idea may seem like a mere application of the theory of rational herds or information cascades in sequential decision-making, initiated by Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992). However, the typical environments studied in that literature share the important feature that a player's payoffs do not directly depend on the actions of others. Since there are no payoff interdependencies, the *only* externality in these models is informational. It is thus straightforward that in these "economic" settings, each player optimally behaves like a backward-looking Bayesian, and has no strategic reason to consider the impact of her choice on any subsequent player's choice. In contrast, an election is a game *with* payoff interdependencies: a voter's payoffs are determined not only by her own action or vote, but also by the votes of those before and after her. Thus, an individual's vote may affect her payoffs in two different ways. First, a vote changes the likelihood of a particular candidate winning in the event that all other votes are tied. Second, a vote can signal information to future voters, and therefore induce different (distributions over) actions of future voters. Given the direct action externalities and the associated incentive to signal information, the optimal choice of a voter is no longer independent of how her vote influences subsequent voters. In particular, a voter may wish to vote *against* the candidate she currently believes to be better, so as to communicate her private information to future voters and/or prevent future voters from herding. Due to this forward-looking incentive, the possibility of rational herds in a sequential voting environment raises significantly different issues from the standard herding literature.

We prove that in our sequential voting setting, there is a Perfect Bayesian equilibrium with *Posterior-Based Voting* (PBV) strategies: Partisans vote for their preferred candidate, and each Neutral voter votes for the candidate who she believes to be better at the time of casting her vote, fully taking into account the information revealed in prior votes. PBV is the appropriate generalization of *sincere voting* (Austen-Smith and Banks, 1996) when moving from simultaneous to sequential voting in incomplete information environments. In voting sincerely, a voter uses all available information—her prior, signal, and observed history of votes in a sequential election—to form her expectation of which candidate is better for her, and votes for this candidate. Importantly, there is no presumption of conditioning on being pivotal. In other words, in PBV, each voter does in fact behave like a backward-looking Bayesian. A main contribution of this paper is to show that this is optimal behavior given that other voters behave in the same way. Importantly, for generic parameters, PBV is a strict equilibrium; thus, it does not rely on the choice of how to resolve indifferences among players, and it is robust to small perturbations of the model. This equilibrium leads to herding by later voters with high probability in large elections, converging to probability one as the population size grows to infinity.

PBV is appealing precisely because it does not require a voter to condition on hypothetical events of being pivotal in deciding how to vote. That PBV is an equilibrium implies that even if a sophisticated voter were to reason in this way, there is no incentive to behave otherwise. There are two reasons why it is noteworthy, and perhaps unexpected, that PBV constitutes an equilibrium in the sequential election: first, as previously noted, the payoff interdependencies in voting may suggest that some complicated forward-looking behavior is necessary in any equilibrium; second, sincere voting is generically not an equilibrium in simultaneous elections, as first noted by Austen-Smith and Banks (1996). The key intuition as to why voting sincerely is an equilibrium in the sequential voting game is that when other players follow PBV, if a deviation were to change the outcome of the election, the subsequent profile of votes can never generate enough public information to outweigh a voter's posterior that is computed from her observed history and private signal. In fact, this logic is so pervasive in the sequential voting environment we consider that voting sincerely is a (generically strict) equilibrium for *all* anonymous and monotonic voting rules. Moreover, so long as voters play the PBV equilibrium, a large class of such so-called *quota rules* are asymptotically equivalent in terms of the ex-ante probability that a particular candidate is elected.

In PBV equilibrium, play is inherently dependent on history, and Neutral voters update their beliefs about which candidate is better based upon the voting history and expectations of partisanship. Accordingly, PBV features rich momentum effects, where the prospects for a candidate can ebb and flow during the course of an election. The notion that such history-dependent voting can be an outcome of a fully rational sequential voting model may be surprising. Some of the early informal literature in political science has invoked assumptions such as a psychological desire to vote for the eventual winner to explain momentum (e.g. Berelson, Lazarsfeld, and McPhee, 1954); this has recently been formalized by Callander (2004a) in a model with an infinite population of voters, each of whom is partially motivated to elect the right candidate, and partially motivated to simply vote for the eventual winner. While his analysis is innovative, and presents interesting comparisons with analogous simultaneous elections (Callander, 2004b), we believe that it is important to establish a non-behaviorally-based benchmark model of momentum, especially because the empirical evidence on whether voters possess conformist motivations is inconclusive.[3]

In an important contribution, Dekel and Piccione (2000) study a model of strategic voting and show that under some assumptions, a class of symmetric equilibria of their simultaneous voting game remain equilibria of the corresponding sequential voting game. Naturally, such equilibria feature history-independent voting. Unfortunately, this result has sometimes been misinterpreted as implying that when a voter conditions on being pivotal in a sequential voting environment, the observed history of votes is irrelevant to her voting decision. Our analysis explicitly shows this is not the case; in particular, in the PBV equilibrium we derive, strategic voters *do* learn from previous voters *and* they act fully rationally, voting as if they are pivotal. We do not view our findings as contradicting Dekel and Piccione (2000); rather, we identify an appealing equilibrium of the sequential election that has no counterpart in simultaneous elections, since it features history-dependent voting. Our main interest in studying PBV is to provide a positive description of behavior that can account for the rich momentum effects exhibited in real world elections.

While the informational model here contributes to the understanding of electoral dynamics, we do not wish to suggest that it captures the whole story. In our model, we abstract from many important institutional details such as campaign finance and media attention.[4] Having said that, it is difficult to shed light on why both financial and media resources are devoted to the first few elections without making specific assumptions about voting behavior. Insofar as the purpose of many elections—especially primaries—is to provide and aggregate information, we believe that these institutional details should be embedded in an informational model similar

[3]See the discussion in Bartels (1988, pp. 108–112). Kenney and Rice (1994) test the strength of various explanations for momentum, including both the preference for conformity theory and an informational theory similar to the one proposed here. Studying the 1988 Republican primaries, they find that voters acted in ways that are consistent with both theories. Examining the results of a national survey conducted during the 1992 Democratic presidential primaries, Mutz (1997) finds strong support for an informational theory where voters learn from the actions of others.

[4]Klumpp and Polborn (2005) offer an analysis of momentum in this vein. They examine a model where two candidates lobby to win primaries by allocating a fixed pie of campaign funds across states. The winner in each state is random, but influenced by the campaign funds allocated to that state. Sequential primaries engender momentum since the winner of an earlier primary has stronger incentives to lobby for later victories.

to the one here. Thus, our work may be viewed as a first step towards understanding the role that these institutions play in voting behavior.

The plan for the remainder of the paper is as follows. Section 2 lays out the model, and Section 3 derives the main results about PBV strategies and equilibrium. We discuss various implications and extensions of our analysis in Section 4. Section 5 concludes. All formal proofs are deferred to the Appendix.

2 Model

We consider a voting game with a finite population of n voters, where n is odd. Voters vote for one of two candidates, L or R, in a fixed sequential order, one at a time. We label the voters $1, \ldots, n$, where without loss of generality, a lower numbered voter votes earlier in the sequence. Each voter observes the entire history of votes when it is her turn to vote. The winner of the election, denoted $W \in \{L, R\}$, is selected by simple majority rule. The state of the world, $\omega \in \{L, R\}$, is unknown, but individuals share a common prior over the possible states, and $\pi > \frac{1}{2}$ is the ex-ante probability of state L. Before voting, each voter i receives a private signal, $s_i \in \{l, r\}$, drawn from a Bernoulli distribution with precision γ (i.e., $\Pr(s_i = l|\omega = L) = \Pr(s_i = r|\omega = R) = \gamma$), with $\gamma > \pi$.[5] Individual signals are drawn independently conditional on the state.

In addition to being privately informed about her signal, a voter also has private information about her preferences: she is either an L-partisan (L_p), a Neutral (N), or an R-partisan (R_p). We denote this *preference type* of voter i by t_i. Each voter's preference type is drawn independently from the same distribution, which assigns probability $\tau_L \in (0, \frac{1}{2})$ to preference type L_p, probability $\tau_R \in (0, \frac{1}{2})$ to R_p, and probability $\tau_N \equiv 1 - \tau_L - \tau_R$ to the Neutral type, N. The preference ordering over candidates is state dependent for Neutrals, but state independent for Partisans. Specifically, payoffs for voter i are defined by the function $u(t_i, W, \omega)$ as follows:

$$\begin{aligned} u(L_p, W, \omega) &= \mathbf{1}_{\{W=L\}} \text{ for } \omega \in \{L, R\} \\ u(R_p, W, \omega) &= \mathbf{1}_{\{W=R\}} \text{ for } \omega \in \{L, R\} \\ u(N, L, L) &= u(N, R, R) = 1 \\ u(N, L, R) &= u(N, R, L) = 0 \end{aligned}$$

Therefore, a voter of preference type C_p ($C \in \{L, R\}$) is a Partisan for candidate C, and desires this candidate to be elected regardless of the state of the world. A Neutral voter, on the other hand, would like to elect candidate $C \in \{L, R\}$ if and only if that candidate is the better one, i.e. if the state $\omega = C$. It is important to observe that an individual vote affects one's payoffs only indirectly through the vote's influence on the winner, W.

[5] This implies that any individual's signal is more informative than the prior. Our analysis will carry over with obvious changes to cases where the signal precision is asymmetric across states of the world.

Denote by $G(\pi, \gamma, \tau_L, \tau_R; n)$ the sequential voting game defined above with prior π, signal precision γ, preference type parameters τ_L and τ_R, and n voters. Throughout the subsequent analysis, we use the term *equilibrium* to mean a (weak) Perfect Bayesian equilibrium of this game (Fudenberg and Tirole, 1991). Let $h^i \in \{L, R\}^{i-1}$ be the realized history of votes when it is voter i's turn to act; denote $h^1 = \phi$. A pure strategy for voter i is a map $v_i : \{L_p, N, R_p\} \times \{L, R\}^{i-1} \times \{l, r\} \to \{L, R\}$. We say that a voter i *votes informatively* following a history h^i if $v_i(N, h^i, l) = L$ and $v_i(N, h^i, r) = R$. The posterior probability that voter i places on state L is denoted by $\mu_i(h^i, s_i)$.

3 Posterior-Based Voting

3.1 Definition and Dynamics

We start the analysis by introducing *Posterior-Based Voting* (PBV) and characterizing its induced dynamics. This allows us to discuss whether such behavior is an equilibrium in Section 3.2. Let $\mathbf{v} = (v_1, \ldots, v_n)$ denote a strategy profile and $\mathbf{v}^i = (v_1, \ldots, v_{i-1})$ denote a profile of strategies for all players preceding i.

Definition 1. *A strategy profile, $\mathbf{v}$, satisfies (or is) Posterior-Based Voting (PBV) if for every voter i, type t_i, history h^i, signal s_i, and for any $W, W' \in \{L, R\}$,*

1. $\mathbb{E}_\omega[u(t_i, W, \omega)|h^i, s_i; \mathbf{v}^i] > \mathbb{E}_\omega[u(t_i, W', \omega)|h^i, s_i; \mathbf{v}^i] \Rightarrow v_i(t_i, h^i, s_i) = W$

2. $\mathbb{E}_\omega[u(t_i, L, \omega)|h^i, s_i; \mathbf{v}^i] = \mathbb{E}_\omega[u(t_i, R, \omega)|h^i, s_i; \mathbf{v}^i] \Rightarrow \begin{cases} v_i(t_i, h^i, l) = L \\ v_i(t_i, h^i, r) = R \end{cases}$

PBV is a property of a strategy *profile*. When we refer to a PBV strategy, we mean a strategy for a player that is part of a PBV profile.

The first part of the definition requires that given the history of votes and her private signal, if a voter believes that electing candidate L (R) will yield strictly higher utility than electing candidate R (L), then she votes for candidate L (R). In other words, in a PBV profile, each voter updates her beliefs about candidates using all currently available information (taking as given the strategies of previous voters), and then votes *sincerely* for a candidate she currently believes to be best for her. Since Partisan voters have a preference ordering over candidates that is independent of the state of the world, the definition immediately implies that Partisans vote for their preferred candidate in a PBV profile, independent of signal and history. Whenever a Neutral voter's posterior is $\mu_i(h^i, s_i) \neq \frac{1}{2}$, she votes for the candidate she believes to be strictly better.

Part two of the definition is a tie-breaking rule. It requires that when a Neutral voter has posterior $\mu_i(h^i, s_i) = \frac{1}{2}$, she vote informatively. In doing so, she reveals her signal to future voters. While we choose this tie-breaking rule to facilitate exposition, it does not play a significant role in our analysis. Any choice of how to break ties only matters for a non-generic constellation of parameters $(\pi, \gamma, \tau_L, \tau_R)$. That is, for a generic tuple, $(\pi, \gamma, \tau_L, \tau_R)$, when PBV is played, it will never be the case that there is a Neutral voter with posterior $\mu_i(h^i, s_i) = \frac{1}{2}$. We defer a formal discussion of this point to Remark 1 in the Appendix. Moreover, our results extend in a straightforward manner to any specification of tie-breaking that is uniform across voters.

The above discussion implies that the behavior of voter i in the PBV profile can be summarized as follows:

$$
\begin{aligned}
v_i(L_p, h^i, s_i) &= L \\
v_i(R_p, h^i, s_i) &= R \\
v_i(N, h^i, s_i) &= \begin{cases} L & \text{if } \mu_i(h^i, s_i) > \frac{1}{2} \text{ or } \{\mu_i(h^i, s_i) = \frac{1}{2} \text{ and } s_i = l\} \\ R & \text{if } \mu_i(h^i, s_i) < \frac{1}{2} \text{ or } \{\mu_i(h^i, s_i) = \frac{1}{2} \text{ and } s_i = r\} \end{cases}
\end{aligned}
$$

PBV is sophisticated insofar as voters infer as much as possible from the past history, taking into account the strategies of preceding players. However, since PBV does not involve any computations regarding the implications of one's vote on future votes, it represents behavior without a forward-looking component. *A priori*, this makes it unclear whether PBV can be an equilibrium. In particular, a rational voter would vote conditioning on being pivotal (affecting the outcome of the election), and therefore should certainly consider how her vote affects the decisions of those after her. We postpone this issue to Section 3.2, instead focussing for the moment on the voting dynamics induced by PBV.

To provide intuition, we start with an informal description of PBV dynamics. Partisans always vote for their preferred candidate, so the matter of interest concerns the behavior of Neutrals. Voter 1, if Neutral, votes informatively, since signals are more informative than the prior. All subsequent Neutral voters face a simple Bayesian inference problem: conditional on the observed history and their private signal, what is the probability that the state is L? Consider $i > 1$ and a history $\tilde{h}^i$ where all preceding Neutrals (are assumed to) have voted informatively, with $\tilde{h}^i$ containing k votes for L and $i - k - 1$ votes for R. For any history, we define the *public likelihood ratio*, $\lambda\left(h^i\right)$, as the ratio of the public belief that the state is L versus state R after the history h^i: $\lambda\left(h^i\right) = \frac{\Pr\left(\omega = L | h^i\right)}{\Pr\left(\omega = R | h^i\right)}$. Therefore, for the history $\tilde{h}^i$,

$$
\lambda\left(\tilde{h}^i\right) = \frac{\pi}{(1-\pi)} \left(\frac{\tau_L + \tau_N \gamma}{\tau_L + \tau_N (1-\gamma)}\right)^k \left(\frac{\tau_R + \tau_N (1-\gamma)}{\tau_R + \tau_N \gamma}\right)^{i-k-1} \tag{1}
$$

The above ratio captures how informative the history $\tilde{h}^i$ is, given the postulated behavior about preceding voters. Since $\gamma > 1 - \gamma$, the above ratio is strictly increasing in k, i.e. seeing a greater number of votes for L strictly raises Neutral voter i's belief that L is the better candidate. Partisanship makes the public history noisy: when $\tau_L \simeq \tau_R \simeq 0$, the ratio is close to its maximum, which is informationally equivalent to voter i having observed k signal l's and

$i-k-1$ signal r's. On the other hand, when $\tau_L \simeq \tau_R \simeq \frac{1}{2}$, the ratio is approximately $\frac{\pi}{1-\pi}$, reflecting a public history only slightly more informative than the prior. The Neutral voter i combines the information from the public history, h^i, with that of her private signal, s_i, to determine her posterior belief, $\mu\left(h^i, s_i\right)$, that the state is L. By Bayes Rule,

$$\frac{\mu\left(h^i, s_i\right)}{1-\mu\left(h^i, s_i\right)} = \lambda\left(h^i\right) \frac{\Pr\left(s_i|\omega = L\right)}{\Pr\left(s_i|\omega = R\right)}$$

Since the signal precision of s_i is γ, it follows that for $\lambda\left(h^i\right) \in \left[\frac{1-\gamma}{\gamma}, \frac{\gamma}{1-\gamma}\right]$, an l signal translates into a posterior belief no less than $\frac{1}{2}$ that the state is L and an r signal translates into a posterior belief no less than $\frac{1}{2}$ that the state is R. However, for $\lambda\left(h^i\right) > \frac{\gamma}{1-\gamma}$, both l and r signals generate posterior beliefs strictly greater than $\frac{1}{2}$ that the state is L, and thus in PBV, a Neutral voter i would vote uninformatively for candidate L. Similarly, for $\lambda\left(h^i\right) < \frac{1-\gamma}{\gamma}$, voter i's posterior favors R regardless of her private signal, and a Neutral i thus votes uninformatively for candidate R.

The behavior of Neutrals in PBV is thus as follows. All Neutrals vote informatively until the public likelihood ratio, $\lambda\left(h^i\right)$, no longer lies in $\left[\frac{1-\gamma}{\gamma}, \frac{\gamma}{1-\gamma}\right]$; when this happens, all Neutrals vote uninformatively for the candidate favored by the posterior — they herd. There are three points to be emphasized about the nature of these herds. First, even after a herd begins for a candidate, Partisans continue to vote for their preferred candidate. Thus, it is always possible to see votes contrary to the herd, and any such contrarian vote is correctly inferred by future voters as having come from a Partisan. Second, at any history where the winner of the election is yet undecided, a herd forming on, say, candidate L, does not immediately imply a victory for L. This is because if all subsequent voters are R-partisans (an event of positive probability), R will in fact be elected. Third, it is possible for a herd to form on a candidate who is *trailing* because the informational content of the voting history is not limited to merely whether a candidate is leading, but also how that candidate is performing relative to the ex-ante distribution of private preferences.

We now turn to a precise and general characterization of behavior and dynamics in PBV. It is convenient to use two state variables that summarize the impact of history on behavior. For any history, h^i, the *vote lead* for candidate L, $\Delta\left(h^i\right)$, is defined recursively as follows:

$$\Delta(h^1) = 0; \text{ for all } i > 1,\ \Delta(h^i) = \Delta\left(h^{i-1}\right) + \left(\mathbf{1}_{\{v_{i-1}=L\}} - \mathbf{1}_{\{v_{i-1}=R\}}\right) \tag{2}$$

The second state variable, called the *phase*, summarizes whether learning is ongoing in the system (denoted phase 0), or has terminated in a herd for one of the candidates (denoted phase L or R). The phase mapping is thus $\Psi : h^i \to \{L, 0, R\}$, and defined by the following transition mapping:

$$\Psi\left(h^1\right)=0;\text{ for all } i>1,\ \Psi\left(h^i\right)=\begin{cases}\Psi\left(h^{i-1}\right) & \text{if } \Psi\left(h^{i-1}\right)\in\{L,R\}\\ L & \text{if } \Psi\left(h^{i-1}\right)=0 \text{ and } \Delta\left(h^i\right)=n_L\left(i\right)\\ R & \text{if } \Psi\left(h^{i-1}\right)=0 \text{ and } \Delta\left(h^i\right)=-n_R\left(i\right)\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

Note that herding phases, $\Psi\in\{L,R\}$, are absorbing. The sequences $n_L(i)$ and $n_R(i)$ in the phase map equation (3) are determined by explicitly considering posteriors, corresponding to our earlier discussion of the public likelihood ratio. For example, assuming that all prior Neutrals voted informatively, $n_L(i)$ is the smallest vote lead for candidate L such that at a history h^i with $\Delta\left(h^i\right)=n_L(i)$, the public history in favor of L outweighs a private signal r. Therefore, the threshold $n_L(i)$ is the unique integer less than or equal to $i-1$ that solves

$$\Pr\left(\omega=L|\Delta\left(h^i\right)=n_L\left(i\right)-2,s_i=r\right)\leq\frac{1}{2}<\Pr\left(\omega=L|\Delta\left(h^i\right)=n_L\left(i\right),s_i=r\right) \tag{4}$$

If it is the case that a history h^i with $\Delta\left(h^i\right)=i-1$ is outweighed by signal r, we set $n_L(i)=i$. Similarly, the threshold $n_R(i)$ is the unique integer less than or equal to i that solves

$$\Pr\left(\omega=L|\Delta\left(h^i\right)=-n_R\left(i\right)+2,s_i=l\right)\geq\frac{1}{2}>\Pr(\omega=L|\Delta\left(h^i\right)=-n_R\left(i\right),s_i=l) \tag{5}$$

where again, implicitly, it is assumed that all prior Neutrals voted informatively. If it is the case that a signal l outweighs even that history h^i where $\Delta\left(h^i\right)=-\left(i-1\right)$, we set $n_R(i)=i$.

We summarize with the following characterization result (all proofs are in the Appendix).

Proposition 1. *Every game $G\left(\pi,\gamma,\tau_L,\tau_R;n\right)$ has a unique PBV strategy profile. For each $i\leq n$, there exist thresholds, $n_L(i)\leq i$ and $n_R(i)\leq i$, such that if voters play PBV in the game $G\left(\pi,\gamma,\tau_L,\tau_R;n\right)$, then a Neutral voter i votes*

1. *informatively if $\Psi\left(h^i\right)=0$;*
2. *uninformatively for $C\in\{L,R\}$ if $\Psi\left(h^i\right)=C$,*

   where Ψ is as defined in (3). The thresholds $n_L(i)$ and $n_R(i)$ are independent of the population size, n.

Plainly, when voters play the PBV strategy profile, a herd develops if and only if there is a history h^i such that $\Psi\left(h^i\right)\neq0$. To study how likely this is, assume without loss of generality that the true state is R. Fixing play according to PBV, the realized path of play is governed by the draw of preference-types and signals. Consequently, the public likelihood ratios can be viewed as a stochastic process, which we denote $\langle\lambda_i\rangle$, where each λ_i is the public likelihood ratio when it is voter i's turn to act. It is well-known (e.g. Smith and Sorensen, 2000) that this stochastic process is a martingale conditional on the true state, R. By the Martingale Convergence Theorem, the process $\langle \lambda_i \rangle$ converges almost surely to a random variable, λ_∞.[6] That is, the public likelihood ratio must eventually converge or settle down if the population were infinite. Since PBV is informative so long as $\Psi = 0$, convergence of the public likelihood ratio requires that $\Psi \in \{L, R\}$ in the limit, i.e. herds eventually occur with probability 1. This intuition underlies the following result for our finite voter game.

Theorem 1. *For every* $(\pi, \gamma, \tau_L, \tau_R)$ *and for every* $\varepsilon > 0$, *there exists* $\overline{n} < \infty$ *such that for all* $n > \overline{n}$, *if voters play PBV, then* $\Pr[\Psi(h^n) \neq 0$ *in* $G(\pi, \gamma, \tau_L, \tau_R; n)] > 1 - \varepsilon$.

3.2 PBV Equilibrium

PBV requires that a voter vote sincerely for who she believes to be better given the available information at the time of casting her vote. We believe that such a behavioral prescription is attractive from a descriptive standpoint: insofar as many voters may simply vote for the candidate they *actually* prefer on the basis of public and private information—as opposed to hypothetically *would* prefer if pivotal—PBV accommodates their behavior. However, the theory would be unattractive if any strategic voter has an incentive to deviate from PBV when all other voters are following it. As noted earlier, since strategic voters have forward-looking incentives in our environment, whereas PBV is explicitly only backward-looking, the strategic optimality of PBV is far from obvious.

We now establish that PBV is optimal behavior for a voter even when she does condition on being pivotal, given that all other voters are following PBV. Consequently, PBV is an equilibrium of the sequential voting game, and therefore accommodates both sincere and strategic voting. In fact, we prove an even stronger result. Say that the *election is undecided* at history h^i if both candidates still have a chance to win the election given the history h^i. An equilibrium is *strict* if conditional on others following their equilibrium strategies, it is uniquely optimal for a voter to follow her equilibrium strategy at any undecided history.[7] We will show that not only is PBV an equilibrium, but moreover, it is generically a strict equilibrium.

To show that PBV is a (strict) equilibrium, the following three behaviors need to be shown as being (uniquely) optimal when all other voters are playing PBV, and the election remains undecided:

(i) Partisans always vote for their preferred candidate;

[6] The careful reader will note that we are discussing an infinite sequence, even though the game consists of a finite number of voters. For the discussion, consider simply extending the sample path of realized likelihood ratios one voter at a time by adding a voter with a new draw of a preference-type and signal (recall these are i.i.d conditional on the true state) who plays PBV.

[7] This definition is slightly non-standard. Usually, a strict equilibrium of a game is one in which a deviation to any other strategy makes a player strictly worse off (Fudenberg and Tirole, 1991). A sequential voting game (with $n \geq 3$) cannot possess any strict equilibria in this sense, because after any history where a candidate has captured a majority of the votes, all actions yield identical payoffs. That is, only histories where the election remains undecided are strategically relevant to voters. Our notion of strictness is therefore the appropriate modification of the usual definition.

(ii) In the herding phase for L (R), Neutrals vote uninformatively for L (R);

(iii) In the learning phase, Neutrals vote informatively.

Throughout the subsequent discussion, we discuss optimality for one voter facing a history where the election is undecided, implicitly assuming that all others are playing PBV. Note that since PBV prescribes Partisans to vote for their preferred candidate independent of history or private signal, every voter who is voting at a history where the election is undecided is pivotal with positive probability. For expositional simplicity, we also restrict attention in our discussion to generic parameters of the game, where it can be shown that conditional on others following PBV, no voter is indifferent at any undecided history. Our formal results and proofs in the Appendix are precise about the role of genericity.

Start by considering the incentives for a Partisan voter. The key observation is that by our earlier characterization (Proposition 1), PBV entails that voting behavior is weakly monotonic in the voting history: by voting L rather than R, a voter can only shift a subsequent player's vote from R to L—if affect it all—but not from L to R; similarly, if she votes R rather than L. Therefore, a Partisan voter cannot influence any future voter in a desirable direction by voting against her preferred candidate. On the other hand, by voting against her preferred candidate, she will be strictly worse off in those events where all other voters are also Partisans, and the votes of all other voters end up exactly tied. These two facts combine to imply that it is strictly optimal for a Partisan to vote for her preferred candidate.[8]

Lemma 1. *If all other players are playing PBV and the election is undecided at the current history, it is strictly optimal for a Partisan to vote for her preferred candidate.*

Next, consider incentives for Neutrals in the herding phase. Suppose that Neutral voter i faces a history where a herd has already begun on candidate L, i.e. $\Psi\left(h^i\right) = L$. Since all subsequent voting after voter i is completely uninformative, conditioning on being pivotal does not change i's posterior beliefs about the state of the world. Consequently, Neutral voter i strictly prefers to vote on the basis of her actual posterior, $\mu_i(h^i, s_i)$. By construction of the phase mapping, $\Psi\left(h^i\right) = L$ implies that $\mu_i\left(h^i, s_i\right) > \frac{1}{2}$. Thus, Neutral voter i strictly prefers to vote for L regardless of her signal. Since the reasoning is symmetric for Neutrals' behavior once an R-herd has already begun, the following holds.

Lemma 2. *If all other players are playing PBV and the election is undecided at the current history, h^i, it is strictly optimal for a Neutral voter i to vote for candidate C if $\Psi(h^i) = C$, for all $C \in \{L, R\}$.*

It remains to establish the main step: that it is optimal for a Neutral voter to vote informatively when votes can still reveal private information, i.e. when in the learning phase,

[8] Interestingly, eliminating weakly-dominated strategies is not sufficient to guarantee that Partisans vote for their preferred candidate, unlike the case of a simultaneous election. In fact, in sequential voting, there exist equilibria (in undominated strategies) where a Partisan voter strictly prefers to vote against her preferred candidate.

$\Psi = 0$. Strategic deviations in the learning phase could significantly influence the behavior of future Neutral voters. With two possible signals and many possible vote leads, establishing that it is optimal for Neutral voters to vote informatively requires showing that numerous incentive compatibility conditions are satisfied. The trade-off faced by Neutral voters can be summarized as follows. By construction of the phase state variable, the public history in the learning phase is not sufficiently informative so as to overturn the private signal received by a Neutral voter. That is, for all h^i where $\Psi\left(h^i\right) = 0$, voter i has posterior $\mu_i(h^i, s_i) \geq \frac{1}{2}$ if $s_i = l$ and $\mu_i(h^i, s_i) \leq \frac{1}{2}$ if $s_i = r$. Therefore, by voting informatively, a Neutral voter i is voting in the direction favored by her posterior. The cost of voting informatively is that she may be pushing future voters closer towards herding. This is costly to a Neutral insofar as herding suppresses the valuable information possessed by later Neutrals. Therefore, when receiving a signal in favor of the leading candidate, a Neutral voter is faced with the trade-off between voting informatively and inducing future Neutral voters to vote informatively. This trade-off is most stark for the Neutral voter i who faces $\Delta\left(h^i\right) = n_L\left(i+1\right) - 1$ (or $-n_R\left(i+1\right) + 1$), since voting for candidate L (or R) in this immediately starts a herd, ending social learning altogether. Would it better to suppress her own information (by voting against her posterior) so as to preserve the value of her successors' information?

We argue that it is strictly optimal for Neutral i to vote informatively even when she triggers a herd. The underlying intuition is that if i's deviation from informative voting changes the outcome of the election, then given PBV by all other voters, there can never be enough information contained in the profile of subsequent votes that makes it worthwhile for i to have voted against her posterior. To see this clearly, we examine the symmetric case where $\tau_L = \tau_R$. It can be verified that this symmetry implies that the thresholds $n_L\left(j\right)$ and $n_R\left(j\right)$ do not vary across voters; instead, they can be denoted simply as constants $n_L > 0$ and $n_R > 0$. Without loss of generality, we consider the case where a Neutral voter i receives signal l and faces history h^i where $\Psi\left(h^i\right) = 0$ and $\Delta\left(h^i\right) = n_L - 1$. The relative strength of the incentives of voter i can be assessed by analyzing the events in which her vote is pivotal, since in any other event, voter i is indifferent between her choices.

Being pivotal in a sequential voting game introduces subtleties relative to a simultaneous voting game. Since by hypothesis all other voters are playing PBV, the impact of voter i's vote can be assessed for any vector of realizations of preference types and signals amongst the future voters. The set of type-signal realizations in which voter i is pivotal, denoted as Piv_i, consist of all those vectors where $v_i = L$ results in L winning the election, whereas $v_i = R$ results in R winning.[9] The hypothesis that the election is as yet undecided implies that $Piv_i \neq \emptyset$. Denote the set of type-signal profiles where $v_i = R$ results in a herd for candidate $C \in \{L, R\}$ as ξ^C; let the set of type-signal profiles where no herd forms after $v_i = R$ be denoted $\tilde{\xi}$. Our argument proceeds by demonstrating that voter i's posterior conditional on being pivotal and on each

[9]Note that there are strategy profiles where i can be pivotal in a way that $v_i = L$ results in R winning, whereas $v_i = R$ results in L winning. This is not possible in PBV because as we noted earlier, PBV features a weak monotonicity of subsequent votes in i's vote.

of the three mutually exclusive and exhaustive events, ξ^L, ξ^R or $\tilde{\xi}$, is greater than $\frac{1}{2}$.[10] This implies that even when conditioning on being pivotal, i strictly prefers to vote L rather than R, fully aware that by voting L, she immediately triggers an L-cascade.

First, consider the pivotal event where an L-cascade begins at some point after $v_i = R$. Since voting is uninformative once the cascade begins, $\Pr\left(\omega = L|\xi^L, Piv_i, h^i, l\right) = \Pr\left(\omega = L|\xi^L, h^i, l\right)$.[11] This probability strictly exceeds $\frac{1}{2}$ since $s_i = l$ and an L-cascade forms only when candidate L commands a lead of at least $n_L - 1$ votes from non-i voters in the learning phase. Therefore, $\Pr\left(\omega = L|\xi^L, Piv_i, h^i, l\right) > \frac{1}{2}$.

Similarly, consider the pivotal event where an R-cascade begins at some point after $v_i = R$. Since voting is uninformative once the cascade begins, $\Pr\left(\omega = L|\xi^R, Piv_i, h^i, l\right) = \Pr\left(\omega = L|\xi^R, h^i, l\right)$. Since $\Delta\left(h^i\right) = n_L - 1$, the event ξ^R can happen only if following $v_i = R$, candidate R subsequently gains a lead of n_R votes in the learning phase. This requires that after i's vote, R receive an additional $n_L + n_R - 2$ votes over L in the learning phase. Since L has a vote lead of $n_L - 1$ prior to i's vote, conditioning on ξ^R in effect reveals a *net* total of $n_R - 1$ votes for R in the learning phase, ignoring i's own vote. By definition of n_R, i's belief given her own signal $s_i = l$ and $n_R - 1$ votes for R in the learning phase is strictly in favor of L. In other words, $\Pr\left(\omega = L|\xi^R, Piv_i, h^i, l\right) > \frac{1}{2}$.

Finally, consider the pivotal event where no cascade occurs after $v_i = R$. This implies that amongst voters $i+1, \ldots n$, candidate R receives at most $n_L + n_R - 2$ votes over candidate L; otherwise, an R-cascade would start. By the same logic as in the earlier case with an R-cascade, it follows that $\Pr\left(\omega = L|\tilde{\xi}, Piv_i, h^i, l\right) > \frac{1}{2}$.

These three steps show that conditional on her observed history, signal, and being pivotal, voter i believes candidate L to be the better candidate with probability strictly greater than $\frac{1}{2}$. Since voting L leads to a strictly higher probability of L winning the election, it is strictly optimal for voter i to vote L, even though such a choice marks the end of the learning phase. This logic is generalized to the case where $\tau_L \neq \tau_R$ in the Appendix. In such cases, the thresholds $n_L(j)$ and $n_R(j)$ vary across voters, but while the non-uniformity introduces challenges, the basic logic of the argument remains.

The above discussion only applies to a voter who immediately starts a cascade by vote informatively. As previously noted, the trade-off between voting informatively and inducing future voters to vote informatively is most stark at the boundaries of the learning phase, hence it can also be established that if the incentive constraints are satisfied (strictly) for histories h^i where $\Delta\left(h^i\right) \in \{n_L(i+1) - 1, -n_R(i+1) + 1\}$, then they are satisfied (strictly) for all other vote leads in the learning phase (see Lemma 5 in the Appendix). The following is the summary result for Neutrals in the learning phase.

[10] The simplification that $\tau_R = \tau_L$ ensures that the posterior conditional on a herd forming for a candidate is invariant to *when* the herd begins.

[11] It is worth being clear that even after conditioning on an L-cascade starting, conditioning on being pivotal can reveal some information about future voters: the point is that because voting is uninformative once a cascade begins, such inferences can only be about voters' preference types, and not about their private signals.

Lemma 3. *If all other players are playing PBV and the election is undecided at the current history, h^i, it is (generically, strictly) optimal for a Neutral voter i to vote informatively when $\Psi\left(h^i\right) = 0$ and $\Delta(h^i) \in \{-n_R\,(i+1)+1, \ldots, n_L\,(i+1)-1\}$.*

Lemmas 1, 2, and 3 combine to imply that PBV is (generically, strictly) optimal for any voter who must vote at a history where the election is undecided, given that all other voters are playing PBV. Our main result follows.

Theorem 2. *The PBV strategy profile is an equilibrium, and generically, is strict.*

There are two points to emphasize about PBV equilibrium. First, its strictness for generic parameters implies that its existence does not rely upon how voter indifference is resolved when the election remains undecided. Given that others are playing PBV, a strategic voter follows PBV not because she is indifferent between or powerless to change the outcome, but rather because deviations yield strictly worse expected payoffs. This also implies that the equilibrium is robust to small perturbations of the model. Second, because Partisan voters always vote for their preferred candidates in the PBV equilibrium, every information set is reached with positive probability. Therefore, off-the-equilibrium-path beliefs play no role in our analysis, and PBV is a Sequential Equilibrium (Kreps and Wilson, 1982).

4 Discussion

4.1 Other Equilibria

The previous section demonstrated that sincere behavior in the form of PBV is an equilibrium of the sequential voting game, and moreover, such behavior leads to momentum effects with high probability in large elections. In this section, we consider other equilibria. Given the size of the strategy space in the current environment, it is difficult to characterize all equilibria; instead, we proceed in two steps. First, we consider the class of *Cut-Point Voting* (CPV) strategy profiles introduced by Callander (2004a). While this class does entail some restrictions, it permits a wide range of interactive behavior between voters. We prove that for generic parameters, any CPV equilibrium leads to herding with large probability in large elections. Thereafter, we discuss history-independent equilibria that mimic outcomes of the simultaneous voting game, contrasting the predictions of these equilibria with those of PBV.

The class of CPV strategy profiles is a broad generalization of PBV. CPV behavior for a Neutral is determined by a combination of the private signal and the weight of the public history. To define a CPV profile, let $\mu\left(h^i\right) \equiv \Pr\left(\omega = L|h^i\right)$, so that $\mu\left(h^i\right)$ denotes the public belief following history h^i.

Definition 2. *A strategy profile, $\mathbf{v}$, is a Cut-Point Voting (CPV) strategy profile if there exist*

$0 \leq \mu_* \leq \mu^* \leq 1$ *such that for every voter* i*, history* h^i*, and signal* s_i*,*

$$v_i\left(N, h^i, s_i\right) = \begin{cases} L & \text{if } \mu\left(h^i\right) > \mu^* \text{ or } \{\mu\left(h^i\right) = \mu_* \text{ and } s_i = l\} \\ R & \text{if } \mu\left(h^i\right) < \mu_* \text{ or } \{\mu\left(h^i\right) = \mu^* \text{ and } s_i = r\} \end{cases}$$

In a CPV strategy profile, Neutrals vote according to their signals alone if and only if the public belief when it is their turn to vote lies within $[\mu_*, \mu^*]$; otherwise, a Neutral votes for one of the candidates independently of her private signal. Denote a CPV profile with belief thresholds μ_* and μ^* as $CPV\left(\mu_*, \mu^*\right)$. These thresholds define the extent to which a CPV profile weighs past history relative to the private signal: $CPV\left(0, 1\right)$ corresponds to informative voting (by Neutrals) where history never influences play, whereas $CPV\left(1-\gamma, \gamma\right)$ corresponds to sincere or posterior-based voting by Neutrals. Similarly, $CPV\left(0, 0\right)$ and $CPV\left(1, 1\right)$ represent strategy profiles where every Neutral votes uninformatively for candidate L and R respectively. Therefore, CPV captures a variety of behavior for Neutrals.

A CPV equilibrium is an equilibrium whose strategy profile is a CPV profile. While we are unable to derive a tight characterization of what non-PBV but CPV profiles—if any—constitute equilibria, we can nevertheless show that generically, large elections lead to herds with high probability within the class of CPV equilibria.

Theorem 3. *For every* $(\pi, \gamma, \tau_L, \tau_R)$ *such that* $\tau_L \neq \tau_R$*, and for every* $\varepsilon > 0$*, there exists* $\overline{n} < \infty$ *such that for all* $n > \overline{n}$*, if voters play a CPV equilibrium,* $\Pr[$*a herd develops in* $G\left(\pi, \gamma, \tau_L, \tau_R; n\right)] > 1 - \varepsilon$*.*

The main step of the argument for Theorem 3 shows that when $\tau_L \neq \tau_R$, any CPV equilibrium in a large election must have belief thresholds μ_* and μ^* that are bounded away from 0 and 1 respectively.[12] This implies the desired result because any such "interior" CPV must result in herds with arbitrarily high probability in sufficiently large elections for the same reason as outlined in the discussion preceding Theorem 1.

To understand the import and limitations of Theorem 3, it is useful to examine the restrictions that CPV imposes on strategy profiles. Every CPV profile is in pure strategies, and embeds a number of monotonicity and symmetry restrictions. While we believe that the monotonicity restrictions are reasonable, it is less clear that symmetry across voters with respect to the belief thresholds is particularly reasonable (or unreasonable) in an asymmetric setting like sequential voting. Dropping this restriction would lead to a more general class of profiles, *asymmetric CPV*, where the belief thresholds can be voter specific. Whether all such asymmetric CPV equilibria involve herding (with high probability in large elections) remains an open question.

The analysis of CPV profiles follows naturally as a generalization of PBV. We now take a

[12] In the knife-edge case where $\tau_L = \tau_R$, informative voting by Neutrals ($CPV\left(0, 1\right)$) and Partisans voting for their preferred candidates is an equilibrium for any population size. However, for any difference in partisanship, however small, this profile is no longer an equilibrium in elections with sufficiently large populations (Lemma 13 in the Appendix).

different perspective on non-PBV equilibria, motivated by Dekel and Piccione (2000). These authors have shown that a class of sequential voting games possess history-independent equilibria that are outcome-equivalent to symmetric mixed equilibria of otherwise identical simultaneous voting games. Even though their result does not formally apply to the setting considered here,[13] their main insight holds and is as follows. For any parameter set $(\pi, \gamma, \tau_L, \tau_R, n)$, the simultaneous voting analog of our model possesses a symmetric equilibrium where Partisans vote their bias, and Neutrals randomize, with signal-dependent probabilities, between voting L and R. For each signal, the randomization probabilities are precisely chosen to offset any asymmetry in Partisanship. Turning to the sequential voting game, consider a strategy profile where, independent of history, every voter plays the same distribution over actions for each private signal as she does in the above construction. Since all voters are acting independently of history, the events in which a voter is pivotal is identical in both the simultaneous and sequential games; therefore, since the profile is an equilibrium of the simultaneous game, it is also an equilibrium of the sequential game. Moreover, it can be shown using the approach of Feddersen and Pesendorfer (1997) that this equilibrium achieves *full information equivalence*, aggregating information efficiently in large elections.

While this result in the spirit of Dekel and Piccione (2000) is clearly an important theoretical benchmark, the descriptive implications of such a history-independent equilibrium appears to be at odds with behavior in real dynamic elections. The history-independent equilibrium hinges on it being commonly known that all voting behavior is unaffected by the history of votes. Were a Neutral voter to admit the possibility that her vote would influence that of future voters, her incentives to play according to the history-independent equilibrium may dissipate.[14] In practice, it seems generally accepted that history (in particular, the vote lead for a candidate) influences voting behavior. Indeed, Bartels (1988) and Popkin (1991) argue that voters keep careful track of how candidates have performed relative to expectations when deliberating how to vote, and that the information provided to voters during the primaries is little more than *horse-race* statistics that describe candidates' performance in preceding states.[15] This suggests that it may be counterfactual to focus on an equilibrium where it is commonly believed that history does not influence behavior.[16]

[13]The presence of partisan voters violates Axiom 2 of their paper.

[14]A voter might admit that the behavior of a future voter is influenced by the public history for various reasons. First, she may believe that the future voter is strategically unsophisticated, and would vote sincerely. Second, there may be higher order uncertainty: if the future voter herself presumes than an even later voter votes on the basis of history, then it may be strategically optimal for the future voter to vote on the basis of history, and so forth.

[15]Mutz (1997) has also shown that voters take cues from the actions of those before them to update their evaluations of candidates, and vote sincerely for that candidate.

[16]While experimental evidence can, and perhaps, should, be brought to bear on the issue, we are hesitant to draw conclusions from the current body of experimental work on sequential voting. In large part this is because the setup of experiments such as those of Morton and Williams (1999) and Battaglini, Morton, and Palfrey (2005) differ importantly from the model developed here. The former authors consider an election with three options; the latter authors consider an election with only three voters, but where voters face a cost of voting, building on the theoretical work of Battaglini (2005). The treatment of Hung and Plott (2001) is the closest to our model, but they consider the case of $\tau_L = \tau_R = 0$, which has special properties that do not hold in general

Furthermore, we believe that the sincerity embodied in PBV is attractive for a positive description. In the history-independent equilibrium of a large voting game, almost every Neutral voter votes against the candidate she actually believes to be better at some history. The rationale for such an action is that the voter recognizes that the actions of future voters are uninfluenced by history, and she is strategically sophisticated so as to condition on the hypothetical event in which her vote is pivotal. To the extent that some voters can not or do not condition on being pivotal, and instead simply vote for the candidate they actually believe to be better at the time of casting their vote, the history-independent equilibrium cannot accommodate their behavior. In contrast, PBV equilibrium is compatible with both sincere and strategic voting.

4.2 Other Voting Rules

Generally, equilibria of elections are sensitive to the choice of voting rule, and this has been illustrated in the case of simultaneous elections by the important contributions of Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1998). Holding fixed a strategy profile, changing the voting rule from simple majority to supermajority or unanimity changes the profiles of votes in which one is pivotal, and can therefore change one's posterior conditional on being pivotal. Moreover, in a simultaneous election, sincere voting requires that Neutrals vote informatively (since $\gamma > \pi$). In large games, this cannot be an equilibrium for all voting rules; indeed, with asymmetric partisanship and sufficiently large populations, sincere voting is not an equilibrium even for a simple majority rule in the simultaneous election. So far, we have established that the appropriate generalization of sincere voting to sequential environments—PBV—is an equilibrium with a simple majority rule for all parameters. In this section, we show that PBV is an equilibrium in the sequential voting game for any voting rule, for all parameters. Furthermore, a large class of voting rules are asymptotically equivalent in terms of the electoral outcome they induce.

We study the class of voting rules termed *q-rules*, where if the fraction of votes for L strictly exceeds some number $q \in [0, 1]$, then L wins the election. Let $G(\pi, \gamma, \tau_L, \tau_R; n, q)$ denote the sequential voting game with parameters $(\pi, \gamma, \tau_L, \tau_R; n)$ where votes are aggregated according to the q-rule. Since the PBV strategy profile is defined independently of the voting rule, for two different rules q and q', PBV generates identical behavior in the games $G(\pi, \gamma, \tau_L, \tau_R; n, q)$ and $G(\pi, \gamma, \tau_L, \tau_R; n, q')$. In fact, with minor modifications to the proof of Theorem 2, the following can be shown.

Theorem 4. *PBV is an equilibrium for any q-rule, and generically, is a strict equilibrium.*

That PBV is an equilibrium for any voting rule and any configuration of parameters is striking. While it is straightforward to see that changing the voting rule leaves the incentives

(see Section 4.3). Nevertheless, we note they find evidence of herding in almost 40 percent of experimental rounds.

of Partisan voters or Neutrals in the herding phases unchanged for any q-rule, the incentives of Neutral voters to vote informatively in the learning phase would seem to be affected because the vote profiles for which one is pivotal differs across voting rules. However, the crucial point about PBV is that even if a voter's deviation in the learning phase changes the outcome, the subsequent profile of votes can never generate enough public information to overturn one's posterior based on the private signal and available public history, regardless of the voting rule. This is because the thresholds for herding in PBV are independent of the voting rule, and the amount of information that can be extracted from future voters' actions is determined by these thresholds.

Theorem 4 raises the question of how changes in the voting rule affect the electoral outcome when voters vote according to PBV. Certainly, different rules may yield different outcomes, where by outcome we mean the mapping from preference type and signal profiles to who wins the elections, which is a well-defined mapping under PBV. However, the following result shows that voting rules can be partitioned into three classes such that all rules within any class are asymptotically *ex-ante* equivalent: they elect the same winner with probability approaching 1 in large voting games.

Theorem 5. *Fix any parameters $\pi, \gamma, \tau_L, \tau_R$, and assume that for any n, q, voters play PBV in the game $G(\cdot; n, q)$. For any $\varepsilon > 0$, there exists $\bar{n}$ such that for all $n > \bar{n}$,*

(a) $|Pr(L \text{ wins in } G(\cdot; n, q) - Pr(L \text{ wins in } G(\cdot; n, q')| < \varepsilon$ for all $q, q' \in (\tau_L, 1 - \tau_R)$;

(b) $\Pr(L \text{ wins in } G(\cdot; n, q)) > 1 - \varepsilon$ for all $q \in [0, \tau_L)$;

(c) $\Pr(L \text{ wins in } G(\cdot; n, q)) < \varepsilon$ for all $q \in (1 - \tau_R, 1]$.

Parts (b) and (c) of Theorem 5 are not surprising given the presence of Partisans: in PBV, the probability with which any voter votes L is at least τ_L and at most $(1 - \tau_R)$. Therefore, in any sufficiently large voting game, L wins with probability approaching 1 if $q < \tau_L$ and loses with probability approaching 1 if $q > (1 - \tau_R)$. The important result is part (a) of Theorem 5: all "interior" voting rules—where outcomes are not determined asymptotically by Partisanship alone—are nevertheless asymptotically equivalent. The intuition relies on the fact that for all such voting rules, once Neutrals begin herding on a candidate, that candidate wins with probability approaching one in large electorates. Therefore, all of these interior rules are asymptotically equivalent once a herd begins on a particular candidate. Asymptotic ex-ante equivalence of these voting rules then follows from the observation that the probability of a herd beginning on a particular candidate is independent of the voting rule (since PBV behavior is defined independently of the voting rule), and by Theorem 1, this probability approaches 1 in large games.[17]

It is interesting to contrast Theorems 4 and 5 with the message of Dekel and Piccione (2000). By showing that (responsive) equilibria of a simultaneous election are outcome-equivalent to

[17] In fact, this argument shows that Theorem 5 can be strengthened to an *ex-post* statement: under PBV, for any realized profile of type-signal vectors, all "interior" voting rules are asymptotically equivalent, and all "extremal" voting rules result in one of the candidates winning with probability approaching 1 in large elections.

equilibria of a voting game with any timing structure, Dekel and Piccione (2000) have illustrated the possibility for strategic behavior to be unaffected by the sequential nature of a voting game. These equilibria are highly sensitive to the voting rule but not to the timing of the game, because in these equilibria, conditioning on being pivotal negates any usefulness from observing the history of votes. Theorem 4 demonstrates the possibility for the opposite effect: given that others vote according to PBV, conditioning on being pivotal does not contain more payoff-relevant information than the public history and one's private signal, regardless of the voting rule. Furthermore, Theorem 5 shows that the PBV equilibrium renders all interior voting rules asymptotically equivalent.[18]

4.3 Pure Common Value Elections

Most of the prior papers on sequential voting games have considered pure common value environments, where every voter is Neutral (e.g. Fey, 2000; Wit, 1997; Callander, 2004a). While we believe that partisanship is a genuine aspect of political economy, this section compares our results with this earlier research.

Fix a set of (π, γ, n), i.e. a prior, signal precision, and population size. Consider a sequence of our sequential voting games where the probability of partisanship, (τ_L, τ_R), is vanishing. It can be checked that for all sufficiently small (τ_L, τ_R), the PBV equilibrium involves an L-herd being triggered when $\Delta\left(h^i\right) = 1$, and an R-herd triggering when $\Delta\left(h^i\right) = -2$ (the asymmetry arises because $\pi > \frac{1}{2}$). Moreover, the limit game with no partisanship ($\tau_L = \tau_R = 0$) also has a PBV equilibrium with these thresholds.

Proposition 2. *Assume $\tau_L = \tau_R = 0$. Then there is an equilibrium where all voters use PBV. In this equilibrium, a transition from the learning phase to the herding phase occurs when $\Delta\left(h^i\right) \in \{1, -2\}$.*

Therefore, the PBV equilibrium correspondence is upper hemi-continuous at $\tau_L = \tau_R = 0$. However, the pure common values election differs in one important respect from the sequential voting game when τ_L and τ_R are strictly positive. Since Partisans vote for their preferred candidates regardless of history in the PBV equilibrium, after a cascade begins, there is positive probability that any future vote may be contrary to the herd. On the other hand, in the PBV equilibrium of the pure common values election, once a herd begins, every vote thereafter is for the leading candidate. Consequently, a vote for the losing candidate is an off-the-equilibrium-path action after a herd has begun. This implies that some of the beliefs that sustain the equilibrium are necessarily off-the-equilibrium-path beliefs, and a theory of "reasonable" beliefs

[18] The equivalence result in Theorem 5 is reminiscent of Gerardi and Yariv (2006). While the similarity is striking and interesting, there are substantive differences. Gerardi and Yariv (2006) consider the impact of communication prior to a simultaneous election, and show that mediated communication or one round of public communication renders an equivalence across all non-unanimity voting rules with respect to all equilibrium outcomes. Our result concerns sequential voting, is specific to PBV and asymptotic populations, but does not require any communication.

now becomes necessary: if voters see a vote going against a herd, how should they interpret it, and given their interpretation, would voters still wish to herd?

A natural place to begin understanding this game would be to investigate the implications of imposing standard beliefs-based refinements for signaling games such as the *Intuitive Criterion*, *D1* (Cho and Kreps, 1987), or *Divinity* (Banks and Sobel, 1987). However, none of these refinements have bite in the pure common values sequential voting game. To see why, consider the even stronger refinement criterion of *Never a Weak Best Response* (Kohlberg and Mertens, 1986). If future voters interpret a deviation from a herd as being equally likely to come from a voter with signal $s_i = l$ as from a voter with signal $s_i = r$, then future voters should not update their beliefs at all based on i's vote, and hence it is a weak best response for voter i to deviate from the herd regardless of her signal. Given this, the belief that a deviation is equally likely to come from either signal-type of voter i survives Never a Weak Best Response, which is the strongest of standard dominance-based belief refinements.

In contrast, the aforementioned papers that consider common value sequential voting (Fey, 2000; Wit, 1997; Callander, 2004a) impose the following belief restriction: if a voter i votes for R once an L-herd has begun, it must be believed that $s_i = r$; similarly, if i votes for L once an R-herd has begun, it must be believed that $s_i = l$. Such a belief restriction implies that it is always possible for a voter to reveal a contrarian signal once in a herd (if she desires to); accordingly, we label this belief restriction as *Perpetual Revelation*. Under this condition, Fey (2000) and Wit (1997) show that because of the signaling motive inherent in common value sequential voting, at least one voter with a signal that opposes the herd would always wish to deviate out of the herd and reveal her signal to future voters. That is, in the pure common value setting, Perpetual Revelation is sufficient to halt momentum by inducing anti-herding for at least one voter.

In our view, it is unclear exactly what the justification for imposing Perpetual Revelation is in the pure common values case, given that it is not implied by standard belief-based refinements. On the other hand, the off-path beliefs of ignoring deviations from herds, which supports the PBV equilibrium of Proposition 2, is justified by the analysis in this paper. This is because in the PBV equilibrium for $\tau_L, \tau_R > 0$, any contrarian vote once a herd has begun must (correctly) be attributed to a Partisan, and hence such a vote reveals no information about the voter's private signal. Since this is true for any $\tau_L, \tau_R > 0$ in PBV, the same belief also holds in the limit PBV equilibrium for $\tau_L = \tau_R = 0$. We do not claim that this is the only reasonable or sensible off-path belief in the pure common values game; our argument is that precluding it—by a priori requiring Perpetual Revelation, for example—lacks justification.

5 Conclusion

This paper has proposed an informational theory of momentum and herding in sequential voting environments. Our model of is that of a binary election where a proportion of the voters seek to elect the better candidate, and the remainder have partisan preferences. The central results are that there exists a generically strict equilibrium which leads to herding with high probability in large elections. Importantly, this equilibrium, PBV, features sincerity where each Neutral voter votes for whichever candidate she thinks is better given all currently available information at the time of casting her vote, whereas Partisans vote for their preferred candidate. Such sincere behavior was shown to be fully rational behavior even for sophisticated voters, given the strategies of all other players. The induced dynamics in this equilibrium captures notions of candidate momentum, including the ebbs, flows, and performance relative to expectations in a dynamic election.

Our results paper raise various issues that deserve further study. We conclude by highlighting some of these.

The discussion in Section 4 concerning the ability to attain full information equivalence may suggest that in terms of aggregating information, sequential voting cannot improve over (the best equilibrium of) simultaneous voting in large elections. However, real world sequential voting mechanisms may feature many benefits over simultaneous counterparts that are outside our model. For example, when thinking about presidential primaries, one natural point of departure is that candidates face greater constraints in campaigning across states that hold primaries on the same day than across states whose primaries are sequenced. This can be modelled formally as a constraint on the informativeness of signals that are obtained by voters who vote simultaneously. If sequencing can increase the informativeness of signals, this provides a rationale for greater information aggregation in sequential voting mechanisms.

Our focus in this paper has been on elections with only two options. Given the nature of the candidate winnowing process in the U.S. presidential primaries, it is important to understand the dynamics of sequential voting with more than two candidates. A particularly intriguing possibility is that an appropriately generalized form of Posterior-Based Voting may be an equilibrium of the current model even when there are three or more candidates. We are currently exploring this idea.

We have also restricted attention here to an environment where voting is entirely sequential, one voter at a time. Though there are elections of this form—for example, roll-call voting mechanisms used in city councils and legislatures—there are many dynamic elections, such as the primaries, that literally feature a mixture of simultaneous and sequential voting. To what extent such games possess equilibria with qualitative features similar to PBV is a significant question for future research.

This paper has abstracted away from the role of institutions, and concentrated on voters as being the sole players. Certainly, in practice, there are other forces involved in dynamic elections, many of which are strategic in nature, such as the media, campaign finance contributors, and so forth. By examining the potential for sequential voting alone to create herding, our model provides a benchmark to understand the role of different institutions in electoral momentum.

Finally, we note that our analysis may have implications for social learning with payoff and information externalities more generally outside the confines of voting. Many environments of economic interest—such as dynamic coordination games, timing of investments, or network choice—feature sequential decision-making, private information that has social value, and payoff interdependencies. Our results suggest that information cascades may survive in various environments despite incentives to reveal or distort one's information to successors.

A Appendix

A.1 Proofs for Section 3.1

We begin with preliminaries that formally construct the thresholds $n_L(i)$ and $n_R(i)$ for each constellation of parameters $(\pi, \gamma, \tau_L, \tau_R)$, and each index i. Define the functions

$$f(\tau_L, \tau_R) \equiv \frac{\tau_L + (1 - \tau_L - \tau_R)\gamma}{\tau_L + (1 - \tau_L - \tau_R)(1 - \gamma)}$$

where the domain is $\tau_L, \tau_R \in \left[0, \frac{1}{2}\right)$. It is straightforward to verify that f strictly exceeds 1 over its domain.

For each positive integer i and any integer k where $|k| < i$ and $i - k$ is odd, define the function $g_i(k) = (f(\tau_L, \tau_R))^k \left(\frac{f(\tau_L,\tau_R)}{f(\tau_R,\tau_L)}\right)^{\frac{i-k-1}{2}}$. Note that for a history h^i where $\Delta\left(h^i\right) = k$ and all prior Neutrals voted informatively and Partisans voted for their preferred candidates, $g_i(k) = \frac{\Pr\left(h^i|\omega=L\right)}{\Pr\left(h^i|\omega=R\right)}$; thus $g_i(k) = \left(\frac{1-\pi}{\pi}\right)\lambda\left(h^i\right)$, as defined in equation (1) in the text. Plainly, $g_i(k)$ is strictly increasing in k.

For a given $(\pi, \gamma, \tau_L, \tau_R)$, define $\{n_L(i)\}_{i=1}^{\infty}$ as follows. For all i such that $g_i(i-1) \le \frac{(1-\pi)\gamma}{\pi(1-\gamma)}$, set $n_L(i) = i$. If $g_i(i-1) > \frac{(1-\pi)\gamma}{\pi(1-\gamma)}$, we shall set $n_L(i)$ to be the unique integer that solves:

$$g_i(n_L(i) - 2) \le \frac{(1-\pi)\gamma}{\pi(1-\gamma)} < g_i(n_L(i)) \tag{6}$$

Since $g_i(-(i-1))$ is strictly less than $\frac{(1-\pi)\gamma}{\pi(1-\gamma)}$, and $g_i(k)$ is strictly increasing in k, a unique solution exists to (6).

Similarly, we define $\{n_R(i)\}_{i=1}^{\infty}$ as follows. For all i such that $g_i(-(i-1)) \ge \frac{(1-\pi)(1-\gamma)}{\pi\gamma}$, set $n_R(i) = i$. If $g_i(-(i-1)) < \frac{(1-\pi)(1-\gamma)}{\pi\gamma}$, set $n_R(i)$ to be the unique integer that solves:

$$g_i(-n_R(i) + 2) \ge \frac{(1-\pi)(1-\gamma)}{\pi\gamma} > g_i(-n_R(i)) \tag{7}$$

As before, since $g_i(k)$ is strictly increasing in k, and $g_i(i-1) = (f(\tau_L, \tau_R))^{i-1} \ge 1 > \frac{(1-\pi)(1-\gamma)}{\pi\gamma}$, a unique solution exists to (7).

We use these values of $n_L(i)$ and $n_R(i)$ to define $\Psi(\cdot)$ as in equation (3) from the text, and turn to the proof of Proposition 1.

Proposition 1 on pp. 9

Proof. The claim is obviously true for Voter 1 as $\Psi\left(h^1\right) = 0 \in (-n_R(1), n_L(1))$, and by construction, a PBV strategy involves a Neutral Voter 1 voting informatively. To proceed by induction, assume that the claim about behavior is true for all Neutral voter $j < i$.

Case 1: $\Psi\left(h^i\right) = 0$: All preceding neutrals have voted informatively. It is straightforward to see that the posterior $\mu\left(h^i, s_i\right) = \mu\left(\tilde{h}^i, s_i\right)$ if $\Delta\left(h^i\right) = \Delta\left(\tilde{h}^i\right)$ and $\Psi\left(h^i\right) = \Psi\left(\tilde{h}^i\right) = 0$ (i.e., so long as all preceding neutrals have voted informatively, only vote lead matters, and not the actual sequence). Thus, we can define $\tilde{\mu}_i\left(\Delta, s_i\right) = \mu\left(h^i, s_i\right)$ where $\Delta = \Delta(h^i)$. By Bayes' rule,

$$\tilde{\mu}_i\left(\Delta, l\right) = \frac{\pi\gamma g_i\left(\Delta\right)}{\pi\gamma g_i\left(\Delta\right) + \left(1-\pi\right)\left(1-\gamma\right)}$$

Simple manipulation shows that $\tilde{\mu}_i\left(\Delta, l\right) \geq \frac{1}{2} \Leftrightarrow g_i\left(\Delta\right) \geq \frac{(1-\pi)(1-\gamma)}{\pi\gamma}$. This latter inequality holds since by hypothesis, $\Psi\left(h^i\right) = 0$, and therefore, $\Delta \geq -n_R\left(i\right) + 1$. If $\tilde{\mu}_i\left(\Delta, l\right) > \frac{1}{2}$, then Condition 1 of the PBV definition requires that Neutral voter i vote L given $s_i = l$; if $\tilde{\mu}_i\left(\Delta, l\right) = \frac{1}{2}$, then Condition 2 of the PBV definition requires that Neutral voter i vote L given $s_i = l$.

Similarly, using Bayes' rule,

$$\tilde{\mu}_i\left(\Delta, r\right) = \frac{\pi\left(1-\gamma\right) g_i\left(\Delta\right)}{\pi\left(1-\gamma\right) g_i\left(\Delta\right) + \left(1-\pi\right)\gamma}$$

Simple manipulation shows that $\tilde{\mu}_i\left(\Delta, r\right) \leq \frac{1}{2} \Leftrightarrow g_i\left(\Delta\right) \leq \frac{(1-\pi)\gamma}{\pi(1-\gamma)}$. The latter inequality holds since by hypothesis, $\Psi\left(h^i\right) = 0$, and therefore, $\Delta \leq n_L - 1$. If $\tilde{\mu}_i\left(\Delta, r\right) < \frac{1}{2}$, then Condition 1 of the PBV definition requires that Neutral voter i vote R given $s_i = r$; if $\tilde{\mu}_i\left(\Delta, r\right)$ then Condition 2 of the PBV definition requires that Neutral voter i vote R given $s_i = r$.

Case 2: $\Psi\left(h^i\right) = L$. Then all Neutrals who voted prior to the first time Ψ took on the value L voted informatively, whereas no voter voted informatively thereafter. Let $j \leq i$ be such that $\Psi\left(h^j\right) = L$ and $\Psi\left(h^{j-1}\right) = 0$; therefore, $\Delta\left(h^j\right) = n_L\left(j\right)$. Then, $\mu\left(h^j, s_j\right) = \tilde{\mu}_j\left(n_L\left(j\right), s_j\right)$. Since all voting after that of $(j-1)$ is uninformative, $\mu\left(h^i, s_i\right) = \mu\left(h^j, s_i\right) = \tilde{\mu}_j\left(n_L\left(j\right), s_i\right)$. A simple variant of the argument in Case 1 implies that $\tilde{\mu}_j\left(n_L\left(j\right), l\right) > \frac{1}{2}$, and therefore Condition 1 of the PBV definition requires that Neutral voter i vote L given $s_i = l$. Consider now $s_i = r$. Since $g_j\left(n_L\left(j\right)\right) > \frac{(1-\pi)\gamma}{\pi(1-\gamma)}$, it follows that $\tilde{\mu}_j\left(n_L\left(j\right), r\right) > \frac{1}{2}$, and therefore Condition 1 of the PBV definition requires that Neutral voter i vote L even following $s_i = r$.

Case 3: $\Psi\left(h^i\right) = R$. Then all Neutrals who voted prior to the first time Ψ took on the value R voted informatively, whereas no voter voted informatively thereafter. Let $j \leq i$ be such that $\Psi\left(h^j\right) = R$ and $\Psi\left(h^{j-1}\right) = 0$; therefore, $\Delta\left(h^j\right) = -n_R\left(j\right)$. Then, $\mu\left(h^j, s_j\right) = \tilde{\mu}_j\left(-n_R\left(j\right), s_j\right)$. Since all voting after that of $(j-1)$ is uninformative, $\mu\left(h^i, s_i\right) = \mu\left(h^j, s_i\right) = \tilde{\mu}_j\left(-n_R\left(j\right), s_i\right)$. A simple variant of the argument in Case 1 implies that $\tilde{\mu}_j\left(-n_R\left(j\right), r\right) < \frac{1}{2}$, and therefore Condition (1) of the PBV definition requires that Neutral voter i vote R given $s_i = r$. Consider now $s_i = l$. Since $g_j\left(-n_R\left(j\right)\right) < \frac{(1-\pi)(1-\gamma)}{\pi\gamma}$, it follows that $\tilde{\mu}_j\left(-n_R\left(j\right), l\right) < \frac{1}{2}$, and therefore Condition 1 of the PBV definition requires that Neutral voter i vote R even following $s_i = l$. $\square$

Remark 1. *As promised in the text, we argue here that the tie-breaking Condition (2) of the PBV definition only matters for a non-generic set of parameters* $(\pi, \gamma, \tau_L, \tau_R)$. *Observe that from the proof of Proposition 1, the posterior of voter* i *having observed a history* h^i *and private signal* s_i *is* $\frac{1}{2}$ *if and only if* $\Psi\left(h^i\right) = 0$ *and* $g_i\left(\Delta\left(h^i\right)\right) \in \left\{\frac{(1-\pi)(1-\gamma)}{\pi\gamma}, \frac{(1-\pi)\gamma}{\pi(1-\gamma)}\right\}$. *For any particular* (π, γ, τ_L), *this occurs for at most a countable collection of* τ_R. *Therefore, for a given* (π, γ), *the set*

$$\Gamma_{\pi,\gamma} \equiv \left\{(\tau_L, \tau_R) \in \left(0, \frac{1}{2}\right)^2 : g_i(\Delta) \in \left\{\frac{(1-\pi)(1-\gamma)}{\pi\gamma}, \frac{(1-\pi)\gamma}{\pi(1-\gamma)}\right\} \text{ for some } i \in \mathbb{Z}^+ \text{ and } |\Delta| \leq i\right\}$$

is isomorphic to a 1-dimensional set. Thus, the need for tie-breaking arises only for a set of parameters $(\pi, \gamma, \tau_L, \tau_R)$ *of (Lebesgue) measure* 0.

Theorem 1 on pp. 10

Proof. The proof consists of two steps: first, we show that there must almost surely be a herd in the limit as the population size $n \to \infty$; second, we show that this implies the finite population statement of the Theorem. Assume without loss of generality that the true state is R. (If the true state is L, one proceeds identically, but using the inverse of the likelihood ratio λ_i).

Step 1: As discussed in the text, by the Martingale Convergence Theorem for non-negative random variables (Billingsley, 1995, pp. 468–469), $\lambda_i \xrightarrow{a.s.} \lambda_\infty$ with *Support* $(\lambda_\infty) \subseteq [0, \infty)$. Define $\bar{\Lambda} \equiv [0, \underline{b}] \cup [\bar{b}, \infty)$ and $\Lambda = [0, \underline{b}) \cup (\bar{b}, \infty)$, where $\underline{b}$ (resp. $\bar{b}$) is the likelihood ratio such that the associated public belief that the state is L causes the posterior upon observing signal l (resp. r) to be exactly $\frac{1}{2}$. Note that by their definitions, $\underline{b} < \frac{1}{2} < \bar{b}$. To prove that there must almost surely be a herd in the limit, it needs to be shown that eventually $\langle\lambda_i\rangle \in \Lambda$ almost surely.[19]

We claim that *Support* $(\lambda_\infty) \subseteq \bar{\Lambda}$. To prove this, fix some $x \notin \bar{\Lambda}$ and suppose towards contradiction that $x \in$ *Support* (λ_∞). Since voting is informative when $\lambda_i = x$, the probability of each vote is continuous in the likelihood ratio around x. Moreover, the updating process on the likelihood ratio following each vote is also continuous around x. Thus, Theorem B.2 of Smith and Sorensen (2000) applies, implying that for both possible votes, either (i) the probability of the vote is 0 when the likelihood ratio is x; or (ii) the updated likelihood ratio following the vote remains x. Since voting is informative, neither of these two is true—contradiction.

The argument is completed by showing that $\Pr(\lambda_\infty \in \Lambda) = 1$. Suppose not, towards contradiction. Then since *Support* $(\lambda_\infty) \subseteq \bar{\Lambda}$, it must be that $\Pr\left(\lambda_\infty \in \left\{\underline{b}, \bar{b}\right\}\right) > 0$. Without loss of generality, assume $\Pr(\lambda_\infty = \underline{b}) > 0$; the argument is analogous if $\Pr\left(\lambda_\infty = \bar{b}\right) > 0$.

[19] To be clear, when we say that $\langle\lambda_i\rangle$ eventually lies (or does not lie) in some set S almost surely, we mean that with probability one there exists some $k < \infty$ that for all $i > k$, $\lambda_k \in (\notin)\ S$.

Observe that if $\lambda_m < \underline{b}$ for some m, then by definition of $\underline{b}$ and PBV, $\lambda_{m+1} = \lambda_m < \underline{b}$ and this sequence of public likelihood ratios converges to a point less than $\underline{b}$. Thus $\Pr(\lambda_\infty = \underline{b}) > 0$ requires that for any $\varepsilon > 0$, eventually $\langle \lambda_i \rangle \in [\underline{b}, \underline{b} + \varepsilon)$ with strictly positive probability. But notice that by the definition of $\underline{b}$, if $\lambda_i = \underline{b}$ then voter i votes informatively under PBV and thus if $\lambda_i = \underline{b}$, either $\lambda_{i+1} < \underline{b}$ (if $v_i = R$) or $\lambda_{i+1} = \frac{1}{2}$ (if $v_i = L$). By continuity of the updating process in the public likelihood ratio on the set $[\underline{b}, \bar{b}]$, it follows that if $\varepsilon > 0$ is chosen small enough, then $\lambda_i \in [\underline{b}, \underline{b} + \varepsilon)$ implies that $\lambda_{i+1} \notin [\underline{b}, \underline{b} + \varepsilon)$. This contradicts the requirement that for any $\varepsilon > 0$, eventually $\langle \lambda_i \rangle \in [\underline{b}, \underline{b} + \varepsilon)$ with strictly positive probability.

Step 2: Since $\lambda_i \xrightarrow{a.s.} \lambda_\infty$, λ_i converges in probability to λ_∞, i.e. for any $\delta, \eta > 0$, there exists $\overline{n} < \infty$ such that for all $n > \overline{n}$, $\Pr(|\lambda_n - \lambda_\infty| \geq \delta) < \eta$. Since $\Pr(\lambda_\infty \in \Lambda) = 1$, for any $\varepsilon > 0$, we can pick $\delta > 0$ small enough such that
$\Pr\left(\lambda_\infty \in [0, \underline{b} - \delta) \cup (\bar{b} + \delta, \infty)\right) > 1 - \frac{\varepsilon}{2}$. Pick $\eta = \frac{\varepsilon}{2}$. With these choices of δ and η, the previous statement implies that there exists $\overline{n} < \infty$ such that for all $n > \overline{n}$, $\Pr(\lambda_n \in \Lambda) > 1 - \varepsilon$, which proves the theorem. $\square$

A.2 Proofs for Section 3.2

We need various intermediate steps to prove the results. Throughout, to prove that PBV is an equilibrium, we assume that the relevant history is undecided since all actions at a decided history yield the same payoffs.

Definition 3. *(Winning Prob.) For a history h^i, let $P\left(\Psi\left(h^i\right), \Delta\left(h^i\right), n - i + 1, \omega\right)$ be the probability with which L wins given the phase $\Psi\left(h^i\right)$, the vote lead $\Delta\left(h^i\right)$, the number of voters who have not yet voted $(n - i + 1)$, and the true state is ω.*

Note that once $\Psi\left(h^i\right) \in \{L, R\}$, all players are voting uninformatively, and therefore, $P\left(\Psi\left(h^i\right), \cdot\right)$ is independent of state. For the subsequent results, let K denote $n - i$, and Δ denote $\Delta\left(h^i\right)$.

Lemma 4. *For all h^i, $P\left(\Psi\left(h^i, L\right), \Delta + 1, K, \omega\right) \geq P\left(\Psi\left(h^i, R\right), \Delta - 1, K, \omega\right)$ for all $\omega \in \{L, R\}$. The inequality is strict if $K > \Delta - 1$.*

Proof. Consider any realized profile of preference types and signals of the remaining K voters given true state ω (conditional on the state, this realization is independent of previous voters' types/signals/votes). In this profile, whenever a voter i votes for L given a vote lead $\Delta - 1$, he would also vote L given a vote lead $\Delta + 1$. Thus, if the type-signal profile is such that L wins given an initial lead of $\Delta - 1$, then L would also win given an initial lead of $\Delta + 1$. Since this applies to an arbitrary type-signal profile (of the remaining K voters, given state ω), it follows that $P\left(\Psi\left(h^i, l\right), \Delta + 1, K, \omega\right) \geq P\left(\Psi\left(h^i, r\right), \Delta - 1, K, \omega\right)$. That the inequality is strict if $K > \Delta - 1$ follows from the fact that with positive probability, the remaining K voters may all be Partisans, with exactly Δ more R-partisans than L-partisans. In such a case, L wins given initial informative vote lead $\Delta + 1$, whereas R wins given initial informative vote lead $\Delta - 1$. $\square$

Lemma 1 on pp. 11

Proof. We will begin by showing that an L-partisan always votes L if others are playing PBV strategies. By voting L, an L-partisan's utility is:

$$\mu\left(h^i,s_i\right)P\left(\Psi\left(h^i,L\right),\Delta+1,K,L\right)+\left(1-\mu\left(h^i,s_i\right)\right)P\left(\Psi\left(h^i,L\right),\Delta+1,K,R\right)$$

If she voted R, her utility is

$$\mu\left(h^i,s_i\right)P\left(\Psi\left(h^i,R\right),\Delta-1,K,L\right)+\left(1-\mu\left(h^i,s_i\right)\right)P\left(\Psi\left(h^i,R\right),\Delta-1,N,R\right)$$

It follows from Lemma 4 that the L-partisan voter i strictly prefers to vote L when the election is undecided (i.e. $K>\Delta-1$).

The same arguments apply *mutatis mutandis* to see that R-partisans strictly prefer to vote R when the election is undecided. □

To show that following PBV is optimal for a Neutral voter (conditional on others following PBV strategies), we need to describe the inferences a Neutral voter makes conditioning on being pivotal. As usual, let a profile of type and signal realizations for all other voters apart from i be denoted

$$(t_{-i},s_{-i})\equiv\left((t_1,s_1),...,(t_{i-1},s_{i-1}),(t_{i+1},s_{i+1}),...,(t_n,s_n)\right)$$

Given that other players are playing PBV, for any realized profile (t_{-i},s_{-i}), i's vote deterministically selects a winner because PBV does not involve mixing. For a vote by voter i, $V_i\in\{L,R\}$, denote the winner of the election $x\left(V_i;(t_{-i},s_{-i})\right)\in\{L,R\}$. Then, denote the event in which voter i is pivotal as $Piv_i=\{(t_{-i},s_{-i}):x\left(L;(t_{-i},s_{-i})\right)\neq x\left(R;(t_{-i},s_{-i})\right)\}$. By arguments identical to Lemma 4, for a given profile (t_{-i},s_{-i}), if a subsequent voter after i votes L following $V_i=R$, then she would also do so following $V_i=L$. Therefore,

$$Piv_i=\{(t_{-i},s_{-i}):x\left(L,(t_{-i},s_{-i})\right)=L\text{ and }x\left(R,(t_{-i},s_{-i})\right)=R\}\tag{8}$$

Let $U\left(V_i|h^i,s_i\right)$ denote a Neutral Voter i's expected utility from action $V\in\{L,R\}$ when she faces a history h^i and has a private signal, s_i. If $\Pr\left(Piv_i|h^i,s_i\right)=0$, then no action is sub-optimal for Voter i. If $\Pr\left(Piv_i|h^i,s_i\right)>0$, i's vote changes her expected utility if and only if her vote is pivotal. Therefore, in such cases,

$$U\left(V|h^i,s_i\right)>U\left(\tilde{V}|h^i,s_i\right)\Leftrightarrow U\left(V|h^i,s_i,Piv_i\right)>U\left(\tilde{V}|h^i,s_i,Piv_i\right)\text{ for }V\neq\tilde{V}$$

It follows from equation (8) that $U\left(L|h^i,s_i,Piv_i\right)=\Pr\left(\omega=L|h^i,s_i,Piv_i\right)$ and $U\left(R|h^i,s_i,Piv_i\right)=1-\Pr\left(\omega=L|h^i,s_i,Piv_i\right)$. Therefore, if $\Pr\left(\omega=L|h^i,l,Piv_i\right)>\frac{1}{2}$, it is strictly optimal for a Neutral Voter i to vote for L, and if $\Pr\left(\omega=L|h^i,l,Piv_i\right)<\frac{1}{2}$, it is strictly optimal for a Neutral Voter i to vote R.

Lemma 2 on pp. 11

Proof. The proof mirrors the intuition laid out in the text. Consider a history, h^i, where $\Psi\left(h^i\right) = L$. Since all future Neutral voters vote uninformatively for L, $\Pr\left(\omega = L|h^i, s_i, Piv_i\right) = \Pr\left(\omega = L|h^i, s_i\right)$, which by construction strictly exceeds $\frac{1}{2}$ for all s_i (since $\Psi\left(h^i\right) = L$). Therefore, a Neutral voter strictly prefers to vote L. An analogous argument applies when $\Psi(h^i) = R$. □

Lemma 3 on pp. 13

The proof proceeds in a series of steps. We shall first use an intermediate lemma (Lemma 5) to show that if the incentive constraints hold for certain voters at certain histories of the learning phase, then they hold for all other possible histories in the learning phase. This simplifies the verification of many incentive constraints to that of a few important constraints. We shall then verify that those constraints also hold in Lemmas 6, 9, and 10.

Lemma 5. *Consider any h^i where $\Psi\left(h^i\right) = 0$ and $\Delta\left(h^i\right) = \Delta$. Then if it is incentive compatible for Neutral Voter $(i+1)$ to vote informatively when $\Delta\left(h^{i+1}\right) \in \{\Delta - 1, \Delta + 1\}$, then it is incentive compatible for Neutral Voter i to vote informatively when $\Delta\left(h^i\right) = \Delta$. Moreover, if the incentive compatibility condition for Neutral Voter $(i+1)$ holds strictly at least in one of the two cases when $\Delta\left(h^{i+1}\right) \in \{\Delta - 1, \Delta + 1\}$, then it holds strictly for Neutral Voter i.*

Proof. We prove that it is optimal for i to vote L given signal $s_i = l$; a similar logic holds for optimality of voting R with signal r. It is necessary and sufficient that

$$\begin{array}{l} \tilde{\mu}_i\left(\Delta, l\right)\left[P\left(0, \Delta + 1, K, L\right) - P\left(0, \Delta - 1, K, L\right)\right] \\ -\left(1 - \tilde{\mu}_i\left(\Delta, l\right)\right)\left[P\left(0, \Delta + 1, K, R\right) - P\left(0, \Delta - 1, K, R\right)\right] \end{array} \geq 0 \tag{9}$$

Define the state-valued functions $p\left(\cdot\right)$ and $q\left(\cdot\right)$

$$\begin{aligned} p\left(\omega\right) &= \begin{cases} \tau_L + \left(1 - \tau_L - \tau_R\right)\gamma & \text{if } \omega = L \\ \tau_L + \left(1 - \tau_L - \tau_R\right)\left(1 - \gamma\right) & \text{if } \omega = R \end{cases} \\ q\left(\omega\right) &= \begin{cases} \tau_R + \left(1 - \tau_L - \tau_R\right)\left(1 - \gamma\right) & \text{if } \omega = L \\ \tau_R + \left(1 - \tau_L - \tau_R\right)\gamma & \text{if } \omega = R \end{cases} \end{aligned}$$

Since voter $i+1$ votes informatively if Neutral (because both $\Delta+1$ and $\Delta-1$ are non-herd leads), the probability that $i+1$ votes L and R in state ω is $p\left(\omega\right)$ and $q\left(\omega\right)$ respectively. Noting the recursive relation

$$P\left(\Psi\left(h^i\right), \Delta, K+1, \omega\right) = p\left(\omega\right) P\left(\Psi\left(h^i, l\right), \Delta + 1, K, \omega\right) + q\left(\omega\right) P\left(\Psi\left(h^i, r\right), \Delta - 1, K, \omega\right)$$

it follows that the above inequality holds if and only if

$$\begin{aligned} 0 \leq\ & \tilde{\mu}_i(\Delta, l) \left[\begin{array}{l} \left(P\left(\Psi\left(h^i, l, l\right), \Delta+2, K-1, L\right) - P(0, \Delta, K-1, L)\right) p(L) \\ + \left(P(0, \Delta, K-1, L) - P\left(\Psi\left(h^i, r, r\right), \Delta-2, K-1, L\right)\right) q(L) \end{array} \right] \\ & - (1-\tilde{\mu}_i(\Delta, l)) \left[\begin{array}{l} \left(P\left(\Psi\left(h^i, l, l\right), \Delta+2, K-1, R\right) - P(0, \Delta, K-1, R)\right) p(R) \\ + \left(P(0, \Delta, K-1, R) - P\left(\Psi\left(h^i, r, r\right), \Delta-2, K-1, R\right)\right) q(R) \end{array} \right] \end{aligned}$$

Dividing by $p(R)(1-\tilde{\mu}_i(\Delta, l))$, the above is equivalent to

$$\begin{aligned} 0 \leq\ & \left(\begin{array}{l} \frac{\tilde{\mu}_i(\Delta,l)}{1-\tilde{\mu}_i(\Delta,l)} \frac{p(L)}{p(R)} \left[P\left(\Psi\left(h^i, l, l\right), \Delta+2, K-1, L\right) - P(0, \Delta, K-1, L)\right] \\ - \left[P\left(\Psi\left(h^i, l, l\right), \Delta+2, K-1, R\right) - P(0, \Delta, K-1, R)\right] \end{array} \right) \\ & + \left(\begin{array}{r} \frac{\tilde{\mu}_i(\Delta,l)}{1-\tilde{\mu}_i(\Delta,l)} \frac{q(L)}{p(R)} \left[P(0, \Delta, K-1, L) - P\left(\Psi\left(h^i, r, r\right), \Delta-2, K-1, L\right)\right] \\ -\frac{q(R)}{p(R)} \left[P(0, \Delta, K-1, R) - P\left(\Psi\left(h^i, r, r\right) \Delta-2, K-1, R\right)\right] \end{array} \right) \end{aligned}$$

We now argue that each of the two lines of the right hand side above is non-negative.

1. Since $\frac{\tilde{\mu}_i(\Delta,l)}{1-\tilde{\mu}_i(\Delta,l)} = \frac{\pi\gamma}{(1-\pi)(1-\gamma)} g_i(\Delta)$ and $\frac{p(L)}{p(R)} = f(\tau_L, \tau_R)$, it follows that

$$\begin{aligned} \frac{\tilde{\mu}_i(\Delta, l)}{1-\tilde{\mu}_i(\Delta, l)} \frac{p(L)}{p(R)} &= \frac{\pi\gamma}{(1-\pi)(1-\gamma)} g_{i+1}(\Delta+1) \\ &= \frac{\tilde{\mu}_{i+1}(\Delta+1, l)}{1-\tilde{\mu}_{i+1}(\Delta+1, l)} \end{aligned}$$

   Since IC holds for voter $i+1$ with vote lead $\Delta+1$, observe that if the election is undecided for $i+1$, $\frac{\tilde{\mu}_{i+1}(\Delta+1,l)}{1-\tilde{\mu}_{i+1}(\Delta+1,l)} \geq \frac{P(\Psi(h^i,l,l),\Delta+2,K-1,R)-P(0,\Delta,K-1,R)}{P(\Psi(h^i,l,l),\Delta+2,K-1,L)-P(0,\Delta,K-1,L)}$, which proves that the first line of the desired right hand side is non-negative. If the election is decided for $i+1$ with vote lead $\Delta+1$, then the first line of the desired right hand side is exactly 0.

2. Using the previous identities,

$$\begin{aligned} \frac{\tilde{\mu}_i(\Delta, l)}{1-\tilde{\mu}_i(\Delta, l)} \frac{q(L)}{q(R)} &= \frac{\pi\gamma}{(1-\pi)(1-\gamma)} g_{i+1}(\Delta-1) \\ &= \frac{\tilde{\mu}_{i+1}(\Delta-1, l)}{1-\tilde{\mu}_{i+1}(\Delta-1, l)} \end{aligned}$$

   Since IC holds for voter $i+1$ with vote lead $\Delta-1$, observe that if the election is undecided for $i-1$, then $\frac{\tilde{\mu}_{i+1}(\Delta-1,l)}{1-\tilde{\mu}_{i+1}(\Delta-1,l)} \frac{q(L)}{q(R)} \geq \frac{P(0,\Delta,K-1,R)-P(\Psi(h^i,r,r),\Delta-2,K-1,R)}{P(0,\Delta,K-1,L)-P(\Psi(h^i,r,r),\Delta-2,K-1,L)}$, and thus the second line of the desired right hand side is non-negative. If the election is decided for $i+1$ with vote lead $\Delta-1$, then the second line of the desired right hand side is exactly 0.

Observe that if incentive compatibility holds strictly for voter $i+1$ in either one of the two cases, then at least one of the two lines of the right hand side is strictly positive, and consequently inequality (9) must hold strictly. □

By the above Lemma, we are left to only check the incentive conditions for a Neutral voter i with undecided history h^i such that $\Psi(h^i) = 0$, but voter $i+1$ will not vote informatively when Neutral if either $v_i = L$ or $v_i = R$. This possibility can be divided into two cases:

1. either i's vote causes the phase to transition into a herding phase; or
2. i is the final voter ($i = n$) and $\Delta(h^n) = 0$.

Lemma 6 below deals with the latter case; Lemmas 9 and 10 concern the former. (Lemmas 7 and 8 are intermediate steps towards Lemma 9.)

Lemma 6. *If there exists a history h^n such that $\Psi(h^n) = 0$ and $\Delta(h^n) = 0$, then it is incentive compatible for Voter n to vote informatively. For generic parameters of the game, the incentive compatibility conditions hold strictly.*

Proof. Since $\Delta(h^n) = 0$ and n is the final voter, $\Pr(\omega = L | h^n, s_n, Piv_n) = \Pr(\omega = L | h^n, s_n)$. Since $\Psi(h^n) = 0$, $\Pr(\omega = L | h^n, l) \geq \frac{1}{2} \geq \Pr(\omega = L | h^n, r)$. Therefore, voting informatively is incentive compatible. Recall from Remark 1 that $\Pr(\omega = L | h^n, s_n) = \frac{1}{2}$ for some $s_n \in \{l, r\}$ only if $(\pi, \gamma, \tau_L, \tau_R)$ is such that $(\tau_L, \tau_R) \in \Gamma_{\pi,\gamma}$, which is a set of (Lebesgue) measure 0. If $(\tau_L, \tau_R) \notin \Gamma_{\pi,\gamma}$, then given that $\Psi(h^n) = 0$, $\Pr(\omega = L | h^n, l) > \frac{1}{2} > \Pr(\omega = L | h^n, r)$, and therefore for generic parameters, voting informatively is strictly optimal for voter n. □

Lemma 7. *Consider any h^i where $\Psi(h^i) = 0$ and $\Delta(h^i) = \Delta$. Then, $P(\Psi(h^i, l), \Delta + 1, K, L) \geq P(\Psi(h^i, l), \Delta + 1, K, R)$ and $P(\Psi(h^i, r), \Delta - 1, K, L) \geq P(\Psi(h^i, r), \Delta - 1, K, R)$ implies $P(0, \Delta, K + 1, L) \geq P(0, \Delta, K + 1, R)$.*

Proof. Simple manipulations yield

$$
\begin{aligned}
& P(0, \Delta, K+1, L) - P(0, \Delta, K+1, R) \\
= \; & p(L) P(\Psi(h^i, l), \Delta + 1, K, L) + q(L) P(\Psi(h^i, r), \Delta - 1, K, L) \\
& - \left[p(R) P(\Psi(h^i, l), \Delta + 1, K, R) + q(R) P(\Psi(h^i, r), \Delta - 1, K, R)\right] \\
\geq \; & p(L) P(\Psi(h^i, l), \Delta + 1, K, L) + q(L) P(\Psi(h^i, r), \Delta - 1, K, L) \\
& - \left[p(L) P(\Psi(h^i, l), \Delta + 1, K, R) + q(L) P(\Psi(h^i, l), \Delta - 1, K, R)\right] \\
= \; & p(L) \left[P(\Psi(h^i, l), \Delta + 1, K, L) - P(\Psi(h^i, l), \Delta + 1, K, R)\right] \\
& + q(L) \left[P(\Psi(h^i, r), \Delta - 1, K, L) - P(\Psi(h^i, r), \Delta - 1, K, R)\right] \\
\geq \; & 0
\end{aligned}
$$

where the first inequality uses the fact that $p(L) \geq p(R)$ and $P(\Delta + 1, K, R) \geq P(\Delta - 1, K, R)$; and the last inequality uses the hypotheses of the Lemma. □

Lemma 8. *For all h^i, $P(\Psi(h^i), \Delta, K + 1, L) \geq P(\Psi(h^i), \Delta, K + 1, R)$.*

Proof. Base Step: The Claim is true when $K = 0$. To see this, first note that Δ must be even for $P\left(\Psi\left(h^i\right), \Delta, 1, \omega\right)$ to be well-defined. If $\Delta \neq 0$ (hence $|\Delta| \geq 2$), then $P\left(\Psi\left(h^i\right), \Delta, 1, L\right) = P\left(\Psi\left(h^i\right), \Delta, 1, R\right)$. For $\Delta = 0$, we have $P(0, 1, L) = p(L) > p(R) = P(0, 1, R)$.

Inductive Step: For any $K \geq 2$, the desired inequality trivially holds if $\Delta \in \{-n_R, n_L\}$ because $P(n_L, K, L) = P(n_L, K, R)$ and $P(-n_R, K, L) = P(-n_R, K, R)$. So it remains to consider only $\Delta \in \{-n_R + 1, \ldots, n_L - 1\}$. Assume inductively that $P(\Delta + 1, K - 1, L) \geq P(\Delta + 1, K - 1, R)$ and $P(\Delta - 1, K - 1, L) \geq P(\Delta - 1, K - 1, R)$. [The Base Step guaranteed this for $K = 2$.] Using the previous Lemma, it follows that $P(\Delta, K, L) \geq P(\Delta, K, R)$ for all $\Delta \in \{-n_R + 1, \ldots, n_L - 1\}$. □

Lemma 9. *Consider history h^i such that $\Delta\left(h^i\right) = n_L(i+1) - 1$ and $\Psi\left(h^i\right) = 0$. Then if all other voters are playing PBV, and a neutral voter i receives signal r, it is strictly optimal for her to vote R. Analogously, if $\Delta\left(h^i\right) = -n_R(i+1) + 1$, and if all other voters are playing PBV, and a neutral voter i receives signal l, it is strictly optimal to vote L.*

Proof. Consider h^i such that $\Delta\left(h^i\right) = n_L(i+1) - 1 = \Delta$, and $\Psi\left(h^i\right) = 0$. For it to be strictly optimal for the voter to vote informatively, it must be that

$$\begin{aligned} & \tilde{\mu}_i(\Delta, r) P(0, \Delta - 1, K, L) + (1 - \tilde{\mu}(\Delta, r))(1 - P(0, \Delta - 1, K, R)) \\ > \; & \tilde{\mu}_i(\Delta, r) P(L, \Delta + 1, K, L) + (1 - \tilde{\mu}_i(\Delta, r))(1 - P(L, \Delta + 1, K, R)) \end{aligned}$$

which is equivalent to

$$\frac{\tilde{\mu}_i(\Delta, r)}{1 - \tilde{\mu}_i(\Delta, r)} < \frac{P(L, \Delta + 1, K, R) - P(0, \Delta - 1, K, R)}{P(L, \Delta + 1, K, L) - P(0, \Delta - 1, K, L)} \tag{10}$$

By Lemma 8, $P(0, \Delta - 1, K, L) \geq P(0, \Delta - 1, K, R)$, and by definition, $P(L, \Delta + 1, K, R) = P(L, \Delta + 1, K, L)$. Therefore, the right-hand side of (10) is bounded below by 1. Since $\Psi\left(h^i\right) = 0$, $\mu\left(h^i, r\right) = \tilde{\mu}_i(\Delta, r) < \frac{1}{2}$, the left-hand side of (10) is strictly less than 1, establishing the strict inequality. An analogous argument applies to prove the case where $\Delta\left(h^i\right) = -n_R(i+1) + 1$ and $s_i = l$. □

Lemma 10. *Consider history h^i such that $\Delta\left(h^i\right) = n_L(i+1) - 1$ and $\Psi\left(h^i\right) = 0$. Then if all other voters are playing PBV, and a neutral voter i receives signal l, it is optimal for her to vote L. Analogously, if $\Delta\left(h^i\right) = -n_R(i+1) + 1$, and if all other voters are playing PBV, and a neutral voter i receives signal r, it is optimal to vote R. For generic parameters of the game, the optimality is strict.*

Proof. Consider the information set where $\Psi\left(h^i\right) = 0$, $\Delta\left(h^i\right) = n_L(i+1) - 1$, and $s_i = l$. By the discussion in the text (p. 27), it suffices to show that $\Pr\left(\omega = L | h^i, l, Piv_i\right) \geq \frac{1}{2}$. For any i, and for any $k > i$, let ξ_k^{Ψ} be the set of types $\{(t_j, s_j)\}_{j \neq i}$ that is consistent with history h^i, induces $\left(\Psi\left(h^{k-1}\right), \Psi\left(h^k\right)\right) = (0, \Psi)$ where $\Psi \in \{L, R\}$ after $v_i = R$, and where i's vote is pivotal. Let $K^{\Psi} = \left\{k > i : \xi_k^{\Psi} \neq \emptyset\right\}$. Denote by ξ_{Δ}^0 the set of types $\{(t_j, s_j)\}_{j \neq i}$ that are consistent with

h^i, induces $\Psi(h^n) = 0$ and $\Delta(h^{n+1}) = \Delta < 0$ after $v_i = R$, and where i's vote is pivotal. Let $K^0_\Delta = \{\Delta : \xi^0_\Delta \neq \emptyset\}$. Observe that since the event $(h^i, Piv_i) = \cup_\Psi \left(\cup_{k\in K^\Psi} \xi^\Psi_k\right) \cup \left(\cup_{\Delta\in K^0_\Delta} \xi^0_\Delta\right)$, by the definition of conditional probability

$$\begin{aligned} \Pr\left(\omega = L|h^i, l, Piv_i\right) &= \sum_{\Psi\in\{L,R\}} \sum_{k\in K^\Psi} \Pr\left(\xi^\Psi_k|h^i, l, Piv_i\right) \Pr\left(\omega = L|\xi^\Psi_k, l\right) \\ &\quad + \sum_{\Delta\in K^0_\Delta} \Pr\left(\xi^0_\Delta|h^i, l, Piv_i\right) \Pr\left(\omega = L|\xi^0_\Delta, l\right) \end{aligned}$$

We shall argue that $\Pr\left(\omega = L|h^i, l, Piv_i\right) \geq \frac{1}{2}$ by showing that $\Pr\left(\omega = L|\xi^L_k, l\right) > \frac{1}{2}$ for each $k \in K^L$, $\Pr\left(\omega = L|\xi^R_k, l\right) \geq \frac{1}{2}$ for each $k \in K^R$, and $\Pr\left(\omega = L|\xi^0_\Delta, l\right) \geq \frac{1}{2}$ for each $\Delta \in K^0_\Delta$.

Consider $k \in K^L$: ξ^L_k denotes a set of signal-type realizations that induce an L-herd after the vote of voter $(k-1)$ (and meet the aforementioned conditions). Since only votes in the learning phase are informative,

$$\Pr\left(\omega = L|\xi^L_k, l\right) = \Pr\left(\omega = L|l, \Psi\left(h^{k-1}\right) = 0, \Delta\left(h^k\right) = n_L(k)\right)$$

Given that $v_i = R$, the informational content of this event is equivalent to a history $\tilde{h}^{k-1}$ where $\Delta\left(\tilde{h}^{k-1}\right) = n_L(k) + 1$, and all Neutrals are assumed to have voted informatively. Therefore,

$$\Pr\left(\omega = L|\xi^L_k, l\right) = \frac{\pi\gamma g_{k-1}\left(n_L(k) + 1\right)}{\pi\gamma g_{k-1}\left(n_L(k) + 1\right) + (1-\pi)(1-\gamma)}$$

Observe that $g_{k-1}(n_L(k) + 1) > g_k(n_L(k)) > \frac{(1-\pi)\gamma}{\pi(1-\gamma)}$. Therefore, $\Pr\left(\omega = L|\xi^\Psi_k, l\right) > \frac{\gamma^2}{\gamma^2+(1-\gamma)^2} > \frac{1}{2}$.

Now consider $k \in K^R$: ξ^R_k denotes a set of signal-type realizations that induce an R-herd after the vote of voter $(k-1)$ (and meet the aforementioned conditions). As before, only votes in the learning phase contain information about the state of the world; thus, $\Pr\left(\omega = L|\xi^R_k, l\right) = \Pr\left(\omega = L|l, \Psi\left(h^{k-1}\right) = 0, \Delta\left(h^k\right) = -n_R(k)\right)$. Given that $v_i = R$, the informational content is equivalent to a history $\tilde{h}^{k-1}$ where $\Delta\left(\tilde{h}^{k-1}\right) = -n_R(k) + 1$, and all neutrals are assumed to have voted informatively. Therefore,

$$\Pr\left(\omega = L|\xi^R_k, l\right) = \frac{\pi\gamma g_{k-1}\left(-n_R(k) + 1\right)}{\pi\gamma g_{k-1}\left(-n_R(k) + 1\right) + (1-\pi)(1-\gamma)}$$

As by assumption, $\Delta\left(h^{k-1}\right) = -n_R(k) + 1$ and $\Psi\left(h^{k-1}\right) = 0$, we have $g_{k-1}\left(-n_R(k) + 1\right) \geq \frac{(1-\pi)(1-\gamma)}{\pi\gamma}$. Therefore, $\Pr\left(\omega = L|\xi^R_k, l\right) \geq \frac{1}{2}$.

Now consider the event $\Delta \in K^0_\Delta$: ξ^0_Δ denotes a set of signal-type realizations that induce no herd and a final vote lead of $\Delta < 0$. Therefore,

$$\Pr\left(\omega = L|\xi^0_\Delta, l\right) = \frac{\pi\gamma g_n(\Delta + 1)}{\pi\gamma g_n(\Delta + 1) + (1-\pi)}$$

Since $\Delta(h^n) \in \{\Delta - 1, \Delta + 1\}$and $\Psi(h^n) = 0$, we have $g_n(\Delta + 1) \geq \frac{(1-\pi)(1-\gamma)}{\pi\gamma}$, and therefore $\Pr\left(\omega = L|\xi^0_\Delta, l\right) \geq \frac{1}{2}$.

We use the above three facts to deduce that $\Pr\left(\omega = L|h^i, l, Piv_i\right) \geq \frac{1}{2}$: observe that

$$\sum_{\Psi \in \{L,R\}, k \in K^\Psi} \Pr\left(\xi^\Psi_k|h^i, l, Piv_i\right) + \sum_{\Delta \in K^0_\Delta} \Pr\left(\xi^0_\Delta|h^i, l, Piv_i\right) = 1$$

Therefore, $\Pr\left(\omega = L|h^i, l, Piv_i\right)$ is a convex combination of numbers that are bounded below by $\frac{1}{2}$.

An analogous argument can be made to ensure optimality at the information set where $\Delta\left(h^i\right) = -n_R(i+1) + 1$ and $s_i = r$.

Let us now explain why the incentive conditions hold strictly for generic parameters of the game. Observe that from our arguments above that indifference arises only if there exists some $k \leq n$ and history h^k such that $\Pr\left(\omega = L|h^k, s_k\right) = \frac{1}{2}$. Recall from Remark 1 that this can hold only if $(\pi, \gamma, \tau_L, \tau_R)$ is such that $(\tau_L, \tau_R) \in \Gamma_{\pi,\gamma}$, which is a set of (Lebesgue) measure 0. If $(\tau_L, \tau_R) \notin \Gamma_{\pi,\gamma}$, then for every k, $\Psi\left(h^k\right) = 0$ implies that $\Pr\left(\omega = L|h^k, l\right) > \frac{1}{2} > \Pr\left(\omega = L|h^k, r\right)$. Therefore, for generic parameters of the game, following PBV is strictly optimal for Voter n regardless of history. $\square$

Lemmas 5, 6, 9, and 10 establish Lemma 3: conditional on all others playing according to PBV, it is optimal for Neutrals to vote informatively in the learning phase. Observe that generic parameters of the game yield strict optimality of the incentive conditions in Lemmas 6 and 10, and therefore, by Lemma 5, all the incentive conditions in the learning phase hold strictly generically.

A.3 Proofs for Section 4

Theorem 3 on pp. 15

We argue through a succession of lemmas that there exist $\overline{\mu}^* < 1$ and $\underline{\mu}_* > 0$ such that when $\tau_L \neq \tau_R$, in a large enough election, a CPV (μ_*, μ^*) is an equilibrium only if $\underline{\mu}_* \leq \mu_* < \mu^* \leq \overline{\mu}^*$. This suffices to prove the Theorem, because then, the arguments of Theorem 1 apply with trivial modifications. Note that in all the lemmas below, it is implicitly assumed when we consider a particular voter's incentives that she is at an undecided history.

For any $CPV(\mu_*, \mu^*)$, we can define threshold sequences $\{\tilde{n}_L(i)\}_{i=i}^\infty$ and $\{\tilde{n}_R(i)\}_{i=i}^\infty$ similarly to $\{n_L(i)\}_{i=i}^\infty$ and $\{n_R(i)\}_{i=i}^\infty$, except using the belief threshold μ^* (resp. μ_*) in place of the PBV threshold γ (resp. $1-\gamma$). That is, for all i such that $g_i(i-1) \leq \frac{(1-\pi)\mu^*}{\pi(1-\mu^*)}$, set $\tilde{n}_L(i) = i$. If $g_i(i-1) > \frac{(1-\pi)\mu^*}{\pi(1-\mu^*)}$, we set $\tilde{n}_L(i)$ to be the unique integer that solves $g_i(\tilde{n}_L(i) - 2) \leq \frac{(1-\pi)\mu^*}{\pi(1-\mu^*)} < g_i(\tilde{n}_L(i))$. For all i such that $g_i(-(i-1)) \geq \frac{(1-\pi)(\mu_*)}{\pi(1-\mu_*)}$, set $\tilde{n}_R(i) = i$. If $g_i(-(i-1)) < \frac{(1-\pi)\mu_*}{\pi(1-\mu_*)}$, set $\tilde{n}_R(i)$ to be the unique integer that solves $g_i(-\tilde{n}_R(i) + 2) \geq \frac{(1-\pi)\mu_*}{\pi(1-\mu_*)} > g_i(-\tilde{n}_R(i))$. Given these thresholds sequences $\tilde{n}_L$ and $\tilde{n}_R$, we define the phase mapping $\tilde{\Psi} : h^i \to \{L, 0, R\}$ in the obvious way that extends the PBV phase mapping Ψ. We state without proof the following generalization of Proposition 1.

Proposition 3. *Fix a parameter set* $(\pi, \gamma, \tau_L, \tau_R, n)$. *For each* $i \leq n$, *if voters play* $CPV(\mu_*, \mu^*)$ *in the game* $G(\pi, \gamma, \tau_{L,}\tau_R; n)$, *there exist sequences* $\{\tilde{n}_L(i)\}_{i=i}^{\infty}$ *and* $\{\tilde{n}_R(i)\}_{i=i}^{\infty}$ *satisfying* $|\tilde{n}_C(i)| \leq i$ *such that a Neutral voter* i *votes*

1. *informatively if* $\tilde{\Psi}\left(h^i\right) = 0$;
2. *uninformatively for* L *if* $\tilde{\Psi}\left(h^i\right) = L$;
3. *uninformatively for* R *if* $\tilde{\Psi}\left(h^i\right) = R$;

   where $\tilde{\Psi}$ *is the phase mapping with respect to* $\tilde{n}_L$ *and* $\tilde{n}_R$. *The thresholds* $\tilde{n}_L(i)$ *and* $\tilde{n}_R(i)$ *do not depend on the population size,* n.

Lemma 11. *There exists* $\bar{\mu}^* < 1$ *and* $\underline{\mu}_* > 0$ *such that in any* $CPV(\mu_*, \mu^*)$,

1. *if* $\mu^* > \bar{\mu}^*$ *then* $\tilde{n}_L(i) > n_L(i)$ *for all* i *such that* $n_L(i) < i$;
2. *if* $\mu_* < \underline{\mu}_*$, *then* $-\tilde{n}_R(i) < -n_R(i)$ *for all* i *such that* $-n_R(i) > -i$.

Proof. We give the argument for part (1); it is similar for part (2). Define $\bar{\mu}^*$ by the equality $\frac{\bar{\mu}^*}{1-\bar{\mu}^*} = \frac{\gamma}{1-\gamma} f(\tau_L, \tau_R) f(\tau_R, \tau_L)$. It is straightforward to compute from the definition of $g_i(\cdot)$ that for any k (such that $|k| < i$ and $i - k$ is odd), $g_i(k-2) f(\tau_L, \tau_R) f(\tau_R, \tau_L) = g_i(k)$. Suppose $\mu^* > \bar{\mu}^*$ and there is some i with $\tilde{n}_L(i) \leq n_L(i)$. By the definitions of $n_L(i)$ and $\tilde{n}_L(i)$, and the monotonicity of $g_i(k)$ in k,

$$\begin{aligned} g_i(n_L(i) - 2) &= g_i(n_L(i)) \left[f(\tau_L, \tau_R) f(\tau_R, \tau_L)\right]^{-1} \\ &\geq g_i(\tilde{n}_L(i)) \left[f(\tau_L, \tau_R) f(\tau_R, \tau_L)\right]^{-1} \\ &> \frac{(1-\pi)\,\mu^*}{\pi(1-\mu^*)} \left[f(\tau_L, \tau_R) f(\tau_R, \tau_L)\right]^{-1} \\ &> \frac{(1-\pi)\,\gamma}{\pi\,(1-\gamma)} \end{aligned}$$

contradicting the definition of $n_L(i)$ which requires that $g_i(n_L(i) - 2) \leq \frac{(1-\pi)\gamma}{\pi(1-\gamma)}$. □

Lemma 12. *If all Neutral voters play according to a CPV profile, it is uniquely optimal for an* L*-partisan to vote* L *and an* R*-partisan to vote* R.

Proof. This follows from the weak monotonicity imposed by CPV; trivial modifications to the argument in Lemma 1 establish this result. □

Lemma 13. *In a large enough election,* $CPV(0,1)$ *is not an equilibrium unless* $\tau_L = \tau_R$.

Proof. Suppose all voters play CPV strategy $(0,1)$. Without loss of generality assume $\tau_L > \tau_R$; the argument is analogous if $\tau_L < \tau_R$. Let $\varsigma_t(n)$ denote the number of voters of preference-type $t \in \{L, R, N\}$ when the electorate size is n. Denote $\tau_N = 1 - \tau_L - \tau_R$. Suppose voter 1 is Neutral and has received signal l. She is pivotal if and only if amongst the other $n-1$ voters, the number of L votes is exactly equal the number of R votes. Let $\varsigma_{N,s}(n)$ denote the number of Neutrals who have received signal $s \in \{l, r\}$. Under the CPV profile $(0,1)$, voter 1 is pivotal if and only if $\varsigma_{N,r}(n) - (\varsigma_{N,l}(n) - 1) = \varsigma_L(n) - \varsigma_R(n)$. By the Weak Law of Large Numbers, for any $\varepsilon > 0$ and any $t \in \{L, R, N\}$, $\lim_{n\to\infty} \Pr\left(\left|\frac{\varsigma_t(n)}{n} - \tau_t\right| < \varepsilon\right) = 1$. Consequently, since $\tau_L > \tau_R$, for any $\varepsilon > 0$ and $k > 0$, there exists $\bar{n}$ such that for all $n > \bar{n}$, $\Pr(\varsigma_L(n) - \varsigma_R(n) > k) > 1 - \varepsilon$. Thus, denoting Piv_1 as the set of preference-type and signal realizations where the Neutral voter 1 with $s_i = l$ is pivotal, we have that for any $\varepsilon > 0$ and $k > 0$, there exists $\bar{n}$ such that for all $n > \bar{n}$, $\Pr(\varsigma_{N,r}(n) - \varsigma_{N,l}(n) > k | Piv_1) > 1 - \varepsilon$. Since $\Pr(\omega = L | \varsigma_{N,r}(n), \varsigma_{N,l}(n))$ is strictly decreasing in $\varsigma_{N,r}(n) - \varsigma_{N,l}(n)$, it follows that by considering k large enough in the previous statement, we can make $\Pr(\omega = L | Piv_1) < \frac{1}{2}$ in large enough elections. Consequently, in large enough elections, voter 1 strictly prefers to vote R when she is Neutral and has received $s_i = l$, which is a deviation from the CPV strategy $(0,1)$. □

Lemma 14. *In a large enough election, CPV (μ_*, μ^*) is not an equilibrium if either $\mu_* > \frac{1}{2}$ or $\mu^* < \pi$.*

Proof. If $\mu_* > \pi$, then the first voter votes uninformatively for R if Neutral, and consequently, all votes are uninformative. Thus, conditioning on being pivotal adds no new information to any voter. Since $\mu_1(h^1, l) > \pi > \frac{1}{2}$ (recall that $h^1 = \phi$), voter 1 has an incentive to deviate from the CPV strategy and vote L if she is Neutral and receives signal $s_1 = l$.

If $\mu_* \in \left(\frac{1}{2}, \pi\right]$, let h^{k+1} be a history of k consecutive R votes. It is straightforward that for some integer $k \geq 1$, $\mu(h^k) \geq \mu_* > \mu(h^{k+1})$. Since an R-herd has started when it is voter $k+1$'s turn to vote, conditioning on being pivotal adds to information to voter $k+1$. Suppose voter $k+1$ is Neutral and receives $s_{k+1} = l$. Then since an R-herd has started, she is supposed to vote R. But since $\mu_{k+1}(h^{k+1}, l) > \mu(h^k) \geq \mu_* > \frac{1}{2}$, she strictly prefers to vote L.

If $\mu^* < \pi$, the argument is analogous to the case of $\mu_* > \pi$, noting that $\mu_1(h^1, r) < \frac{1}{2}$ because $\gamma > \pi$. □

Lemma 15. *In a large enough election, CPV $(\mu_*, 1)$ is not an equilibrium for any $\mu_* \in (0, \pi]$.*

Proof. Suppose *CPV* $(\mu_*, 1)$ with $\mu_* \in (0, \pi]$ is an equilibrium. Consider a Neutral voter m with signal $s_m = r$ and history h^m such that $\mu(h^m) \geq \mu_*$ but $\mu(h^{m+1}) < \mu_*$ following $v_m = R$. (To see that such a configuration can arise in a large enough election, consider a sequence of consecutive R votes by all voters.) Voter m is supposed to vote R in the equilibrium. We will show that she strictly prefers a deviation to voting L in a large enough election.

Claim 1: If the true state is R, then following $v_m = L$, the probability of an R-herd converges to 1 as the electorate size $n \to \infty$. *Proof*: Recall that the likelihood ratio stochastic process

$\lambda_i \overset{a.s.}{\to} \lambda_\infty$ (where the domain can be taken as $i = m+1, m+2, \ldots$). Since voter i votes informatively if and only if $\lambda_i \geq \frac{\mu_*}{1-\mu_*}$, the argument used in proving Theorem 1 allows us to conclude that $Support\left(\lambda_\infty\right) \subseteq \left[0, \frac{\mu_*}{1-\mu_*}\right]$ and $\Pr\left(\lambda_\infty = \frac{\mu_*}{1-\mu_*}\right) = 0$. Consequently, there is a herd on R eventually almost surely in state R.

Claim 2: $\Pr\left(Piv_m|\omega = R\right)$ converges to 0 as the electorate size $n \to \infty$. *Proof*: To be explicit, we use superscripts to denote the electorate size n, e.g. we write Piv_m^n instead of Piv_m. Denote

$$
\begin{aligned}
X^n &= \{(t_{-m}, s_{-m}) \in Piv_m : L\text{-herd after } v_m = L,\ R\text{-herd after } v_m = R\} \\
Y^n &= \{(t_{-m}, s_{-m}) \in Piv_m : \text{no herd after } v_m = L,\ R\text{-herd after } v_m = R\} \\
Z^n &= \{(t_{-m}, s_{-m}) \in Piv_m : R\text{-herd after } v_m = L \text{ and } v_m = R\}
\end{aligned}
$$

We have $Piv_m^n = X^n \cup Y^n \cup Z^n$; hence it suffices to show that $\Pr\left(X^n\right) \to 0$, $\Pr\left(Y^n\right) \to 0$, and $\Pr\left(Z^n\right) \to 0$. That $\Pr\left(X^n\right) \to 0$ and $\Pr\left(Y^n\right) \to 0$ follows straightforwardly from Claim 1. To show that $\Pr\left(Z^n\right) \to 0$, let Ψ_k^n denote the phase after voter k has voted, i.e. when it is voter $k+1$'s turn to vote. For any n, consider the set of $\{(t_j, s_j)\}_{j=m+1}^n$ such that after $v_m = L$, $\Psi_n^n \neq L$; denote this set Ξ^n. Partition this into the sets that induce $\Psi_n^n = 0$ and $\Psi_n^n = R$, denoted $\Xi^{n,0}$ and $\Xi^{n,R}$ respectively. Clearly, $Z^n \subseteq \Xi^{n,R}$. For any ε, for large enough n, regardless of m's vote, $\Pr\left(\Psi_n^n = 0\right) < \varepsilon$ by Claim 1, and thus, $\Pr\left(\Xi^{n,0}\right) < \varepsilon$. Now consider any $n' > n$. $Z^{n'} \subseteq \Xi^n$ because if there is a L-herd following $v_m = L$ with electorate size n, there cannot be an R-herd following $v_m = L$ with electorate size n'. Thus, $\Pr\left(Z^{n'}\right) = \Pr\left(\Xi^{n,0}\right)\Pr\left(Z^{n'}|\Xi^{n,0}\right) + \Pr\left(\Xi^{n,R}\right)\Pr\left(Z^{n'}|\Xi_n^R\right) < \varepsilon + \Pr\left(\Xi^{n,R}\right)\Pr\left(Z^{n'}|\Xi_n^R\right)$ for large enough n. We have $\Pr\left(Z^{n'}|\Xi^{n,R}\right) = \dfrac{\Pr\left(Z^{n'} \cap \Xi^{n,R}\right)}{\Pr\left(\Xi^{n,R}\right)}$. It is straightforward to see that $\Pr\left(Z^{n'} \cap \Xi^{n,R}\right) \to 0$ as $n' \to \infty$, using the fact that $\tau_L < 1 - \tau_L$ and invoking the Weak Law of Large Numbers similarly to Lemma 13. Note that $\Pr\left(\Xi^{n,R}\right)$ is bounded away from 0 because if sufficiently many voters immediately after m are R-partisans, then an R-herd will start regardless of m's vote. This proves that $\Pr\left(Z^{n'}\right) \to 0$.

Claim 3: If the true state is L, then following $v_m = L$, the probability that L wins is bounded away from 0 as the electorate size $n \to \infty$. *Proof*: Define $\xi\left(h^i\right) = \frac{\Pr\left(\omega = R|h^i\right)}{\Pr\left(\omega = L|h^i\right)}$; this generates a stochastic process $\langle \xi_i \rangle$ ($i = m+1, m+2, \ldots$) which is a martingale conditional on state L, and thus $\langle \xi_i \rangle \overset{a.s.}{\to} \xi_\infty$. Note that $\xi_{m+1} < \frac{1-\mu_*}{\mu_*}$ since $\mu\left(h^m\right) \geq \mu^*$ and $v_m = L$. Since voter i votes informatively if and only if $\xi_i \in \left(0, \frac{1-\mu_*}{\mu_*}\right]$, the argument used in proving Theorem 1 allows us to conclude that $Support\left(\xi_\infty\right) \subseteq \{0\} \cup \left[\frac{1-\mu_*}{\mu_*}, \infty\right)$ and $\Pr\left(\xi_\infty = \frac{1-\mu_*}{\mu_*}\right) = 0$. Suppose towards contradiction that $0 \notin Support\left(\xi_\infty\right)$. This implies $\mathbb{E}\left[\xi_\infty\right] > \frac{1-\mu_*}{\mu_*}$. By Fatou's Lemma (Billingsley, 1995, p. 209), $\mathbb{E}\left[\xi_\infty\right] \leq \lim_{n\to\infty} \mathbb{E}\left[\xi_n\right]$; since for any $n \geq m+1$, $\mathbb{E}\left[\xi_n\right] = \xi_{m+1}$, we have $\frac{1-\mu_*}{\mu_*} < \mathbb{E}\left[\xi_\infty\right] \leq \xi_{m+1} < \frac{1-\mu_*}{\mu_*}$, a contradiction. Thus, $0 \in Support\left(\xi_\infty\right)$, and it must be that $\Pr\left(\xi_\infty = 0\right) > 0$. The claim follows from the observation that for any history sequence where $\xi_i\left(h^i\right) \to 0$ it must be that $\Delta\left(h^i\right) \to \infty$.

Consider the expected utility for voter m from voting R or L respectively, conditional on being pivotal: $EU_m\left(v_m = R|Piv_m\right) = \Pr\left(\omega = R|Piv_m\right)$ and $EU_m\left(v_m = L|Piv_m\right) = \Pr\left(\omega = L|Piv_m\right)$. Thus, she strictly prefers to vote L if and only if $\Pr\left(\omega = L|Piv_m\right) > \Pr\left(\omega = R|Piv_m\right)$, or equivalently, if and only if $\Pr\left(Piv_m|\omega = L\right) > \Pr\left(Piv_m|\omega = R\right)\frac{1-\mu_m(h^m,r)}{\mu_m(h^m,r)}$. By Claim 2, $\Pr\left(Piv_m|\omega = R\right)$ converges to 0 as electorate grows. On the other hand, $\Pr\left(Piv_m|\omega = L\right)$ is bounded away from 0, because by Claim 3, the probability that L wins following $v_m = L$ is bounded away from 0, whereas if $v_m = R$, a R-herd starts and thus the probability that R wins converges to 1 as the electorate size grows. Therefore, in a large enough election, $\Pr\left(Piv_m|\omega = L\right) > \Pr\left(Piv_m|\omega = R\right)\frac{1-\mu_m(h^m,r)}{\mu_m(h^m,r)}$, and it is strictly optimal for m to vote L following his signal $s_m = r$, which is a deviation from the CPV strategy. $\square$

Lemma 16. *In a large enough election, $CPV\left(0, \mu^*\right)$ is not an equilibrium for any $\mu^* \in [\pi, 1)$.*

Proof. Analogous to Lemma 15, it can be shown here that in a large enough election there is a voter who when Neutral is supposed to vote L with signal l, but strictly prefers to vote R. $\square$

Lemma 17. *In a large enough election, $CPV\left(\mu_*, \mu^*\right)$ is not an equilibrium if $\mu^* \in (\bar{\mu}^*, 1)$ and $\mu_* \in (0, \frac{1}{2}]$.*

Proof. Fix an equilibrium $CPV\left(\mu_*, \mu^*\right)$ with $\mu^* \in (\bar{\mu}^*, 1)$ and $\mu_* \in (0, \frac{1}{2}]$. By Lemma 11, $\tilde{n}_L(i) > n_L(i)$ for all i. Consider a Neutral voter m with signal $s_m = r$ and history h^m such that $\mu\left(h^m\right) \geq \mu_*$ but $\mu\left(h^{m+1}\right) < \mu_*$ following $v_m = R$. (To see that such a configuration can arise in a large enough election, consider a sequence of consecutive R votes by all voters.) Voter m is supposed to vote R in the equilibrium. We will show that she strictly prefers a deviation to voting L in a large enough election.

First, note that by following the argument of Theorem 1, it is straightforward to show that regardless of m's vote, a herd arises with arbitrarily high probability when the electorate size n is sufficiently large. Define X^n, Y^n, and Z^n as in Lemma 15, where n indexes the electorate size. Plainly, $\Pr\left(Y^n\right) \to 0$. The argument of Claim 2 in Lemma 15 implies with obvious modifications that $\Pr\left(Z^n\right) \to 0$. Finally, $\Pr\left(X^n\right) \nrightarrow 0$ because there exists $m' > m$ such that if $v_i = L$ for all $i \in \{m+1, \ldots, m'\}$, then $\Psi^n_{m'} = L$, and $\Pr\left(v_i = L \text{ for all } i \in \{m+1, \ldots m'\}\right) \geq \left(\tau_L\right)^{m'-m} > 0$. Since $Piv^n_m = X^n \cup Y^n \cup Z^n$, we conclude that as $n \to \infty$, $\Pr\left(X^n|Piv^n_m\right) \to 1$, whereas $\Pr\left(Y^n|Piv^n_m\right) \to 0$ and $\Pr\left(Z^n|Piv^n_m\right) \to 0$. Consequently, for any $\varepsilon > 0$, there exists $\bar{n}$ such that for all $n > \bar{n}$,

$$\left|EU_m\left(v_m = L|X^n, s_m = r\right) - EU_m\left(v_m = L|Piv^n_m, s_m = r\right)\right| < \varepsilon$$

and

$$\left|EU_m\left(v_m = R|X^n, s_m = r\right) - EU_m\left(v_m = R|Piv^n_m, s_m = r\right)\right| < \varepsilon$$

Therefore, it suffices to show that for any $n > m$,

$$EU_m\left(v_m = L|X^n, s_m = r\right) > EU_m\left(v_m = R|X^n, s_m = r\right),$$

or equivalently, $\Pr\left(\omega = L|X^n, s_m = r\right) > \Pr\left(\omega = R|X^n, s_m = r\right)$. For any $k \in \{m+1, \ldots, n\}$, define

$$X_k^n = \left\{(t_{-m}, s_{-m}) \in Piv_m^n : \ \Psi_{k-1}^n = 0 \text{ but } \Psi_k^n = L \text{ after } v_m = L, \ \Psi_n^n \text{ after } v_m = R\right\}$$

Clearly, this requires $\tilde{n}_L(k+1) < k+1$. For $i \neq j$, $X_i^n \cap X_j^n = \emptyset$, but $X^n = \cup_{k=m+1}^n X_k^n$, and thus $\Pr\left(\omega|X^n, s_m = r\right) = \cup_{k=m+1}^n \Pr\left(\omega|X_k^n, s_m = r\right)\Pr\left(X_k^n|X^n\right)$. It therefore suffices to show that for any $k \in \{m+1, \ldots, n\}$, $\Pr\left(\omega = L|X_k^n, s_m = r\right) > \Pr\left(\omega = R|X_k^n, s_m = r\right)$. Given that $v_m = L$, the informational content of X_k^n is equivalent to a history h^{k+1} where $\Delta\left(h^{k+1}\right) = \tilde{n}_L(k+1) - 2$, and all neutrals are assumed to have voted informatively. Therefore,

$$\Pr\left(\omega = L|X_k^n, s_m = r\right) = \frac{\pi\gamma g_{k+1}\left(\tilde{n}_L(k+1) - 2\right)}{\pi\gamma g_{k+1}\left(\tilde{n}_L(k+1) - 2\right) + (1-\pi)(1-\gamma)}$$

Since $\tilde{n}_L(k+1) < k+1$ and $\tilde{n}_L(i) > n_L(i)$ for all i, it must be that $\tilde{n}_L(k+1) - 2 \geq n_L(k+1)$. Consequently,

$$\begin{aligned}\Pr\left(\omega = L|X_k^n, s_m = r\right) &\geq \frac{\pi\gamma g_{k+1}\left(n_L(k+1)\right)}{\pi\gamma g_{k+1}\left(n_L(k+1)\right) + (1-\pi)(1-\gamma)} \\ &> \frac{1}{2}\end{aligned}$$

where the second inequality is by the definition of $n_L(k+1)$. □

Lemma 18. *In a large enough election, CPV (μ_*, μ^*) is not an equilibrium if $\mu^* \in [\pi, 1)$ and $\mu_* \in (0, \underline{\mu}_*)$.*

Proof. Analogous to Lemma 17, it can be shown here that in a large enough election there is a voter who when Neutral is supposed to vote L with signal l, but strictly prefers to vote R. □

Theorem 4 on pp. 17

Proof. The result follows from minor modifications of Theorem 2; in particular, modifying Lemma 6 to $\Psi(h^n) = 0$ and $\Delta(h^n) \in \{\lfloor qn \rfloor, \lceil qn \rceil\}$. □

Theorem 5 on pp. 18

Proof. We shall consider a PBV strategy profile and consider two threshold rules q and q' where $\tau_L < q < q' < 1 - \tau_R$. Given a profile of n votes, let S_n denote the total number of votes cast in favor of L.

Pick $\varepsilon > 0$. From Theorem 1, we know that there exists $\bar{k}$ such that for all $k \geq \bar{k}$, $\Pr\left(\Psi\left(h^k\right) = L\right) + \Pr\left(\Psi\left(h^k\right) = R\right) > 1 - \frac{\varepsilon}{2}$. Pick any $k \geq \bar{k}$. By the Weak Law of Large Numbers, for every $\kappa > 0$, $\lim_{n\to\infty} \Pr\left(\left|\frac{S_n}{n} - (1-\tau_R)\right| < \kappa | \Psi\left(h^k\right) = L\right) = 1$ and $\lim_{n\to\infty} \Pr\left(\left|\frac{S_n}{n} - \tau_L\right| < \kappa | \Psi\left(h^k\right) = R\right) = 1$. Pick some $\kappa < \min\{(1-\tau_R) - q', q - \tau_L\}$. There exists some $\bar{n} > k$ such that for all $n \geq \bar{n}$, $\Pr\left(\left|\frac{S_n}{n} - (1-\tau_R)\right| < \kappa | \Psi\left(h^k\right) = L\right) > 1 - \frac{\varepsilon}{2}$ and $\Pr\left(\left|\frac{S_n}{n} - \tau_L\right| < \kappa | \Psi\left(h^k\right) = R\right) > 1 - \frac{\varepsilon}{2}$. Observe that by the choice of κ, $\left|\frac{S_n}{n} - (1-\tau_R)\right| < \kappa$ implies that L wins under both rules q and q' whereas $\left|\frac{S_n}{n} - \tau_L\right| < \kappa$ implies that L loses under both rules q and q'. For any $n \geq \bar{n}$, we have

$$
\begin{aligned}
& \left|Pr(\text{L wins in } G\left(\pi, \gamma, \tau_L, \tau_R; n, q\right) - Pr(\text{L wins in } G\left(\pi, \gamma, \tau_L, \tau_R; n, q'\right)\right| \\
< & \left|\sum_{x\in\{L,R\}} \Pr\left(\Psi\left(h^k\right) = x\right) \Pr\left(\frac{S_n}{n} \in \left(q, q'\right] | \Psi\left(h^k\right) = x\right)\right| + \frac{\varepsilon}{2} \\
< & \ \varepsilon
\end{aligned}
$$

which proves part (a) of the Theorem.

For part (b), consider any $q < \tau_L$. The probability with which a voter votes L is at least τ_L. Therefore, invoking the Weak Law of Large Numbers, $\lim_{n\to\infty} \Pr\left(\frac{S_n}{n} < q\right) = 0$. The argument is analogous for part (c). □

Proposition 2 on pp. 19

Proof. We shall begin by describing the PBV strategy profile and then proceed to establish that it is an equilibrium. Observe that for $\tau_L = \tau_R = 0$, for every i, $g_i(k) = \left(\frac{\gamma}{1-\gamma}\right)^k$. As $\frac{\gamma}{1-\gamma} > \frac{(1-\pi)\gamma}{\pi(1-\gamma)} > 1 > \frac{1-\gamma}{\gamma} > \frac{(1-\pi)(1-\gamma)}{\pi\gamma} > \left(\frac{1-\gamma}{\gamma}\right)^2$. Therefore,

$$
\left(n_L(i), n_R(i)\right) = \begin{cases} (1,3) & \text{if } i \text{ is even} \\ (2,2) & \text{if } i \text{ is odd} \end{cases}
$$

Therefore, PBV prescribes that if $\Psi\left(h^{i-1}\right) = 0$, and $\Delta\left(h^i\right) \in \{1, -2\}$, then $\Psi\left(h^i\right) \neq 0$. To show that this strategy profile is an equilibrium, we shall consider the incentives in the learning and herding phases separately.

Observe that if $\Psi\left(h^i\right) = L(R)$, voting for $R(L)$ occurs with zero-probability on the path of play. We shall consider a belief-restriction that specifies that any off-path vote is simply ignored and does not affect the public belief. Given this belief-restriction, it can be shown using Theorem 2 (particularly Lemmas 2, 3) that PBV is an equilibrium. □

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