Sequentially Optimal Mechanisms[*]

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Abstract

This paper characterizes the revenue maximizing allocation mechanism in a two-period model under non-commitment. A risk neutral seller has one object to sell and faces a risk neutral buyer whose valuation is private information. The seller has all the bargaining power; she designs an institution to sell the object at t=0 but cannot commit to not change the institution at t=1 if trade does not occur at t=0. We show that the optimal mechanism is to post a price in each period. A methodological contribution of the paper is to develop a procedure to characterize the optimal dynamic incentive schemes under non-commitment in asymmetric information environments where the agent's type is drawn from a continuum. *Keywords: mechanism design, optimal auctions, sequential rationality. JEL Classification Codes: C72, D44, D82.*

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1 Introduction

The bargaining under incomplete information literature[1] acknowledges that, if it is common knowledge that gains of trade exist, parties cannot credibly commit to stop negotiating at a point where no agreement is reached. It examines possible outcomes of negotiations between individuals under the assumption that players make deterministic offers. Often the right to make offers is assigned to one of the negotiating parties. Suppose that the uninformed one makes the offers. Would it be beneficial for her instead of making a take-it-or-leave-it offer at each period, to employ more sophisticated bargaining procedures? Would that possibility allow her to learn the other party's private information faster? What is the optimal from the uninformed party's point of view, negotiating process? In the optimal auction literature, (see Myerson (1981) and Riley and Samuelson (1981)), the seller is free to employ any institution to sell the object but commits never to propose a different mechanism in case no trade takes place. This excludes the possibility of employing a mechanism in the future that may perform better. In other words, the optimal auction is characterized under the restriction that the seller may behave in a non-credible way. This assumption of commitment is often far-fetched.[2] In this paper we relax this assumption and derive the optimal mechanism.

To illustrate the situation let us look at the following scenario. A risk neutral seller faces a risk neutral buyer whose valuation is private information. At $t = 0$ the seller proposes the ex-ante revenue maximizing mechanism under commitment. This consists of posting a price. The buyer announces his valuation to the seller and if it is above the seller's posted price, he obtains the object and pays the price. Suppose that the buyer announces a valuation below the price. No trade takes place. This procedure is optimal *given* that the seller can commit to not try to sell the item using a different institution in a subsequent period. But if the object remains unsold, it is not sequentially rational for the seller to tie her hands. At date 1 the seller knows that there exist gains from trade but they were not realized because the price she posted was above the buyer's valuation. If the seller behaves sequentially rationally she will try to sell the item at $t = 1$ using a different mechanism that maximizes revenue from that point on, which clearly changes the buyer's strategic considerations at $t = 0$. The buyer at $t = 0$ may try to convince the seller that he has a low valuation. What does the revenue maximizing mechanism look like in this case? Does the seller offer a set of lotteries at period $t = 0$ or does she simply post a price? Does the seller use a mechanism in the first period that allows her to infer with precision the type of the buyer, hoping that she can use her sharper estimate to extract the buyer's surplus in the second period? Or is it too costly in terms of expected revenue to do so?

In this paper we look at a seller who faces a buyer whose valuation is private information and is drawn from a continuum. There are 2 periods. At the beginning of each period the

[1] See, for instance, Sobel and Takahashi (1983) and Fudenberg, Levine and Tirole (1985).

[2] Real world examples about the inability of the sellers to commit can be found in McAfee and Vincent (1997).

seller proposes an institution to sell the object. If the object is sold in the first period the game ends, otherwise the seller returns the next period and offers a new mechanism. The game ends after 2 periods even if the object remains unsold. We allow the seller to employ any institution she wishes, and show that posting a price in each period is optimal. This is our main result. It is the first complete characterization of the optimal mechanism under non-commitment in an asymmetric information environment, where the agent's type is drawn from a continuum. Another contribution of this work is to provide a method to characterize the optimal dynamic incentive schemes in such environments.

The early papers on dynamic mechanism design, (Freixas, Guesnerie and Tirole (1985), FGT, Laffont and Tirole (1988)), LT, establish that under non-commitment the principal cannot appeal to the standard revelation principal in order to characterize the optimal mechanism. This makes the characterization of the optimal contract very difficult.[3] For this reason FGT (1985) consider the optimal incentive schemes among the class of *linear* incentive schemes. LT (1988) consider arbitrary schemes but examine only special classes of equilibria, namely pooling and partition equilibria. A remarkable result is derived in the recent paper by Bester and Strausz (2001), BS. They show that when the principal faces *one* agent whose type space is *finite,* she can, without loss of generality, restrict attention to mechanisms where the message space has the same cardinality as the type space. As BS illustrate, in order to find the optimal mechanism one has to check which incentive compatibility constraints are binding. In an environment with limited commitment, constraints may be binding 'upwards' and 'downwards'. Even if one could obtain an analog of the BS result for the continuum type case, it does not seem straightforward to generalize the procedure of checking which IC are binding. In this paper we provide a method for solving for the optimal dynamic scheme under non-commitment for the case that the type space is a continuum. The case of a continuum of types raises difficulties that do not arise in the finite type case. For instance, the support of the principal's posterior beliefs can be arbitrarily complicated, (whereas in the case where one starts with a finite type space, the type space at the beginning of a subsequent period is again finite). A by-product of our analysis is the characterization of the optimal mechanism for the case that the agent's type is arbitrary. To my knowledge this problem has not been addressed before in the literature.

We now provide a brief description of how we obtained our result. The procedure used in this paper in order to characterize the optimal allocation mechanism under non-commitment does not rely on any version of the revelation principle. It can be summarized as follows. Given a belief system μ, a strategy of the seller, σ_S, and a strategy of the buyer, σ_B, implement an allocation rule $p(\sigma, \mu)(v)$, abbreviated to p, and a payment rule $x(\sigma, \mu)(v)$, abbreviated to x. A solution concept imposes restrictions on (σ, μ), which in turn translate to restrictions on p and x. We start by looking at the restrictions imposed on the allocation rule p by requiring (σ, μ) to be a Bayes-Nash Equilibrium, BNE, of the

[3] See the discussion in Laffont and Tirole (1993), Ch. 9, and Salanie (1998), Ch.6.

game. We show that if σ_B is a best response to σ_S, then p is increasing in v, and we can write the seller's expected revenue solely as a *linear* function of p. Subsequently we impose the further requirement that (σ, μ) be sequentially rational, and provide necessary conditions that an allocation rule p satisfies if it is implemented at a Perfect Bayesian Equilibrium, PBE. In order to do so, we study the seller's problem at the beginning of the final period of the game. Since the game ends after that period, the seller's problem at the beginning of the final period of the game is isomorphic to a static problem under commitment. The difference is that the support of the seller's posterior beliefs can be very complicated. We present a method to solve static mechanism design problems when the type space is arbitrarily complicated. Given this result, we provide necessary conditions that an allocation rule satisfies if it is implemented at a PBE. Our objective is to find the optimal allocation rule p^* among all $PBE-$implementable ones. We show that the allocation rule that maximizes the seller's expected discounted revenue among all $PBE-$implementable ones, can be implemented by a PBE of the game where the seller posts a price in each period. It follows that if the seller behaves sequentially rationally, then the institution that maximizes expected revenue is a sequence of posted prices. The method used in order to derive this result was inspired by Riley and Zeckhauser (1983). From our analysis it follows that working with mechanisms with arbitrary message spaces can be as simple as working with direct revelation mechanisms.

Hart and Tirole (1988), HT, analyze a similar model in a finite-horizon framework under non-commitment and commitment and renegotiation. In the non-commitment case the seller's strategy consists of a sequence of *prices*. Our model differs from the one in HT in that we consider a continuum of types and in that we allow the seller to employ arbitrary mechanisms. McAfee and Vincent (1997) examine sequentially optimal auctions under the assumption that the seller's strategy is a sequence of *reservation prices*, and the buyers follow a stationary strategy.

The model studied in this paper also captures the situation of a monopolist who faces a continuum of buyers whose valuation is known. This is the standard framework considered in the durable-good-monopolist literature, (Bulow (1982), Stokey (1981), Gul-Sonnenschein-Wilson (1986)). In those papers the seller posts a price in each period. The possibility of using other institutions is not investigated. The durable-good monopolist literature examines the equilibrium *price* dynamics when the monopolist cannot sign binding contracts with consumers about the future level of prices. We show that this is optimal, even though the seller may use any other institution she wishes. In other words, our results verify that allowing the seller to simply post a price in each period does not entail any loss in terms of expected revenue. Previous work has assumed that the monopolist's - or the uninformed party's- strategy is to post a price and the problem is to find what price to post. We provide a justification for posted prices; even though arbitrarily complicated procedures may be used, posted price selling is the optimal strategy in the sense that it maximizes the seller's revenue.

Summarizing, the literature either studies the mechanism design problem under the assumption of commitment or, acknowledges the impossibility of commitment for a fixed institution, (a price in the durable goods monopoly literature and a reserve price in the sequentially optimal auctions literature), and searches for the optimal price (reserve price) path under the assumption that the seller behaves sequentially rationally. In this paper we study the sale model of a good under non-commitment and we *characterize* the optimal mechanism. The analysis resembles the one of a bargaining model with incomplete information, in which the uninformed party makes the offers. In our model the seller instead of making deterministic offers in each period, she "offers" a game form.

We now provide an outline of the paper. The environment under consideration is described in Section 2. Section 3 outlines our method for characterizing the optimal mechanism under non-commitment. The main analysis and results of this work can be found in Section 4, which is the core of the paper. Section 5 contains some observations regarding revenue comparisons relative to the commitment case and an illustrative example. Concluding remarks are in Section 6.

2 The Environment

A seller owns one object. Her valuation for the object is normalized to zero. She faces one buyer whose valuation v is private information. We use V to denote the set of all possible valuations of the buyer. It is taken to be $V = [a, b]$ for $0 \leq a < b < \infty$. Time is discrete and the game lasts two periods, $t = 0, 1$. Let $f : [a, b] \to \mathbb{R}_{++}$, continuous denote the probability density function of the buyer's valuation. All elements of the game except the realization of the buyer's valuation are common knowledge. Both the seller and the buyer are risk neutral. We use δ to denote the common discount factor. The seller's goal is to maximize expected discounted revenue. The buyer aims to maximize expected surplus.

Definition 1 *A* mechanism $M = (S, g)$ *consists of a set of actions S available to the buyer and an outcome function $g : S \to [0, 1] \times \mathbb{R}_+$.*

Suppose that the seller proposes $M = (S, g)$. When the buyer accepts M and chooses an $s \in S$, the outcome specified via g is a probability that he obtains the good, r, and an expected payment z. A point $(r, z) \in [0, 1] \times \mathbb{R}_+$ will be called a *contract.* Rejecting a mechanism leads to the legal status quo, which is the contract $(0, 0)$. We assume that the action "reject" is always available.

Remark 1 *For our purposes two mechanisms that lead to the same set of contracts are equivalent. Sometimes we will describe a mechanism by the set of contracts that it leads to.*

The assumption of non-commitment, asserts that the seller and the buyer at t cannot commit to anything about their interaction at $t + 1$. We now argue that under non-commitment, the mechanisms under consideration are without loss of generality. In each period the seller and the buyer play a game. This game results, using our terminology, to a set of possible contracts which is a subset of $[0,1] \times \mathbb{R}_+$. Since there is no commitment, the seller and the buyer cannot make any agreement in period t regarding the terms of their future transactions. Their $t-$period interaction results in a set of probability-payment pairs. Suppose that we allowed the seller, apart from proposing M, to send herself messages while M was being played. This game would again lead to a set of possible probability-payment pairs. The buyer's strategy within the $t-$period game would specify for each v which actions the buyer would choose given various messages of the seller and so forth. This situation can be replicated by some mechanism among the class that we are considering. Of course, one could analyze the problem under different assumptions regarding the commitment power of the seller; for instance, one can examine the optimal renegotiation-proof mechanisms. Under those different assumptions, the mechanisms considered here may not be without loss of generality.

Timing.

- At the beginning of period $t = 0$ nature determines the valuation of the buyer. Subsequently the seller designs a mechanism $M_0 = (S_0, g_0)$. The buyer observes M_0 and decides whether to participate in this mechanism. If the buyer rejects M_0 he does not get the object and he pays zero. Otherwise the buyer picks $s \in S_0$ and the outcome is $g_0(s)$. If the item is transferred, then the game ends, else we move on to period $t = 1$.

- At period $t = 1$ the seller designs a new mechanism $M_1 = (S_1, g_1)$. The buyer observes M_1. If the buyer rejects M_1 he does not get the object and he pays zero; otherwise the outcome is determined by g_1. The game terminates at the end of period $t = 1$.

Let $h_S^0 = 0$ be the history after chance's move; we denote by $A_S(h_S^0)$ the set of the seller's possible period-0 actions and by $A_B(h_B^0)$ the buyer's possible period-0 actions. More generally, we use $A_i(h_i^t)$ to denote the set of player $i's$ possible actions, where h_i^t represents the history of moves before player $i's$ move in period t, excluding chance's move. The action of the buyer at t is denoted by s_t.

The information sets of the buyer consist of simply one node since the buyer observes the move of nature and the moves of the seller. On the other hand, since the seller does not observe the realization of the buyer's valuation, her information set in period t is identified with an element of H_S^t, where H_S^t is the set of all feasible histories at date t.

Let $\mathcal{M}$ denote the set of all possible mechanisms. A strategy for the seller, σ_S, is a sequence of maps from H_S^t to $\mathcal{M}$. A strategy, σ_B, for the buyer is a sequence of maps from $[a,b] \times H_B^t$ to $A_B(h_B^t)$. A strategy profile $\sigma = (\sigma_i)_{i=S,B}$, specifies a strategy for each player. A belief system, μ, maps H_S^t to the set of probability distributions over $[a,b]$.

Our aim is to characterize the maximum expected revenue that the seller can guarantee at a PBE. As usual we require that strategies yield a BNE, not only for the whole game, but also for the continuation game that starts at t after each history where trade has not occurred up to t. Let $f(v|h_S^t)$ denote the seller's beliefs about the buyer's valuation when the history is h_S^t, $t = 0, 1$. For $t = 0$ $f(v|h_S^0) = f(v)$, that is, the seller has correct prior beliefs.

A *Perfect Bayesian Equilibrium, (PBE),* is a strategy profile, σ, and a belief system, μ, that satisfy:

1. For all $v \in [a, b]$, s_1 maximizes the buyer's payoff at $t = 1$.

2. Given $f(.|h_S^1)$ and the buyer's strategy, M_1 maximizes expected revenue for the seller.

3. For all $v \in [a, b]$, s_0 maximizes the buyer's payoff given $t = 1$ actions.

4. M_0 maximizes the seller's expected revenue given subsequent actions.

5. $f(.|h_S^1)$ is derived from f given h_S^1 using Bayes' rule whenever possible.

We sometimes simplify the notation by setting $f(v|h_S^1) = f_1(v)$ in what follows. As usual, condition 2 requires that the seller choose at $t = 1$ an optimal mechanism relative to her posterior beliefs even off the equilibrium path.

3 The Methodology

In this section we provide an outline of the method we use to characterize the optimal mechanism. First recall that in dynamic settings under non-commitment the seller cannot appeal to the standard the revelation principal in order to characterize the optimal mechanism at $t = 0$. To see why, suppose that at period zero the seller employs a direct revelation mechanism, the buyer has claimed to have valuation v, and according to this mechanism no trade takes place. If the seller behaves sequentially rationally, she will try to sell the object at $t = 1$ using a different mechanism. And in the case that the buyer has revealed his true valuation at $t = 0$, the seller has complete information at $t = 1$. She can therefore use this information to extract all the surplus from the buyer. In this situation the buyer will have an incentive to manipulate the seller's beliefs. One would expect that he will not always reveal his valuation truthfully at the beginning of the relationship. The seller, since she does not have commitment power, cannot play the role of the "machine" that exogenously specifies the direct revelation game that implements an equilibrium of some general game.

Suppose that the seller could appeal to the revelation principle in order to choose M_0. Then, because according to the revelation principle truth telling is an equilibrium, for each $v \in [a, b]$ the buyer's strategy specifies a different action[4] at $t = 0$ (separation). One can prove a result similar to that in Proposition 1 in Laffont and Tirole (1988), which states that

[4] When the seller offers a DRM, the buyer's action is a claim about his type.

there exists no PBE where separation occurs at $t = 0$. This result provides a justification for why the seller cannot employ the standard revelation principle in the environment under consideration in order to choose M_0.

Allocation Rules and Payment Rules

Our method of characterizing the PBE that is associated with the highest revenue for the seller, relies on the observation that an arbitrary assessment, (σ, μ), implements an allocation rule p and a payment rule x. We first provide a couple of examples to illustrate this.

Example 1 *Take* $V = [0, 1]$. *Suppose that the seller posts a price of* 0.5 *at* $t = 0$ *and if no trade takes place at* $t = 0$, *(which happens only when the buyer rejects* M_0), *she posts a price of* 0.4 *at* $t = 1$. *More formally, at t=0 the seller proposes*

$$M_0 = \{(0, 0), (1, 0.5)\}.$$

If the buyer chooses $(0, 0)$ *trade will not take place at* $t = 0$. *At this history the seller at* $t = 1$ *proposes*

$$M_1 = \{(0, 0), (1, 0.4)\} \ .$$

The buyer's strategy, (specified only up to the path actions), is as follows: for $v \in [0, 0.4]$ *reject* M_0 *at* $t = 0$ *and reject* M_1 *at* $t = 1$; *for* $v \in [0.4, \bar{v}]$ *reject* M_0 *at* $t = 0$ *and pay* 0.4 *at* $t = 1$; *and for* $v \in (\bar{v}, 1]$ *pay* 0.5 *at* $t = 0$. *From the ex-ante point of view, given the strategy profile under consideration, the expected discounted probability that the buyer obtains the object and the expected discounted payment are*

$$\begin{aligned} p(v) = 0;\ x(v) = 0\ \ &\textit{for } v \in [0, 0, 4] \\ p(v) = \delta;\ x(v) = 0.4\delta\ \ &\textit{for } v \in (0.4, \bar{v}] \\ p(v) = 1;\ x(v) = 0.5\ &\textit{for } v \in (\bar{v}, 1]. \end{aligned}$$

Example 2 *Suppose now that the seller's strategy is as in the previous example, with the difference that* $(1, 0.5)$ *in* M_0 *is replaced by* $(0.8, 0.5)$. *Now the seller's strategy must also specify what mechanism the seller will employ at* $t = 1$ *in the event that the buyer chooses* $(0.8, 0.5)$ *at* $t = 0$ *but he does not obtain the object; this mechanism is given by*

$$\tilde{M}_1 = \{(0, 0), (0.3, 0.1)\}.$$

The buyer's strategy, (specified up to the path actions), is as follows: for $v \in [0, 0.4]$ *reject* M_0 *at* $t = 0$ *and* M_1 *at* $t = 1$; *for* $v \in (0.4, \bar{v}_1]$ *reject* M_0 *at* $t = 0$ *and pay* 0.4 *at* $t = 1$; *for* $v \in (\bar{v}_1, \bar{v}_2]$ *choose* $(0.8, 0.5)$ *at* $t = 0$ *and reject* $\tilde{M}_1$ *at* $t = 1$ *and finally for* $v \in (\bar{v}_2, 1]$

choose $(0.8, 0.5)$ *at* $t = 0$ *and* $(0.3, 0.1)$ *at* $t = 1$. *The expected discounted probability and the expected discounted payment implemented by this strategy profile are given by*

$$p(v) = 0;\ x(v) = 0 \ \textit{for}\ v \in [0, 0, 4]$$
$$p(v) = \delta;\ x(v) = 0.4\delta \ \textit{for}\ v \in (0.4, \bar{v}_1]$$
$$p(v) = 0.8;\ x(v) = 0.5 \ \textit{for}\ v \in (\bar{v}_1, \bar{v}_2]$$
$$p(v) = 0.8 + (1 - 0.8) \cdot 0.3 \cdot \delta;\ x(v) = 0.5 + (1 - 0.8) \cdot \delta \cdot 0.2 \ \textit{for}\ v \in (\bar{v}_2, 1].$$

As illustrated in the above examples, an assessment[5] (σ, μ) leads to a set of compounded lotteries: $[p(\sigma, \mu)(v), x(\sigma, \mu)(v)]$. This is the set of expected discounted outcomes of the game given (σ, μ). The rule $p(\sigma, \mu)(v)$, sometimes abbreviated as $p(v)$, maps the type space to probabilities and denotes the expected, discounted probability that a $v-$type buyer will obtain the object given (σ, μ). We will call this function as *allocation rule.* It is formally defined as

$$p(\sigma, \mu)(v) = E\left[\sum_{t=0}^{1} \delta^t 1_{\{\text{buyer obtains the object at } t\}} |(\sigma, \mu), v\right].$$

Allocation rules will play a central role in our analysis. It is possible that different strategy profiles lead to the same allocation rule.

The rule $x(\sigma, \mu)(v)$, sometimes abbreviated as $x(v)$, maps the type space into $\mathbb{R}_+$ and we will call it *payment rule*. It is formally defined as

$$x(\sigma, \mu)(v) = E\left[\sum_{t=0}^{1} \delta^t 1_{\{\text{buyer obtains the object at } t\}} \cdot \{\text{payment at } t\} |(\sigma, \mu), v\right].$$

Probabilities and transfers are in *expectation and discounted*.

A solution concept imposes restrictions on (σ, μ) which, in turn, translate to restrictions on the allocation and on the payment rule. We start by exploring restrictions imposed on the allocation rule and on the payment rule by requiring (σ, μ) to be a Bayes-Nash Equilibrium of the game. We show that if the buyer's strategy is a best response to the seller's strategy, p is increasing in v. Given this result we use standard techniques and write the seller's expected revenue solely as a *linear* function of p. In the next step we derive necessary conditions that an allocation rule satisfies, if it is implemented by an assessment that consists a PBE of the game. Finally we characterize the allocation rule p^* that is the optimal among all $PBE-$implementable ones.

Let us now compare our method with the approach that relies on the standard revelation principle. The revelation principle states that all allocation and payment rules implemented by an assessment that is a BNE of a game, can be implemented by a direct revelation mechanism where truth telling is an equilibrium. One usually derives necessary

[5] We need to include the belief system in the arguments of p and x because it is part of the equilibrium concepts we will examine.

and sufficient conditions satisfied by p and x at a truthful equilibrium. In this paper we are interested in $PBE-$implementable payment and allocation rules. We allow the seller to propose mechanisms with arbitrary message spaces and focus on the outcomes of the game. We start by arbitrary assessments (σ, μ) and examine how equilibrium restrictions translate into properties of the allocation rules. We derive only necessary conditions that allocation rules satisfy if they are implemented at a PBE of the game, but this turns out to be enough for the characterization of the PBE that is associated with maximal revenue for the seller.

This paper demonstrates that mechanism design under non-commitment is not an intractable problem, even if one works with mechanisms with arbitrary message spaces. As long as one cares for the players' payoff from the ex-ante point of view this approach seems as straightforward as working with direct revelation mechanisms.

4 The Optimal Mechanism Under Non-Commitment

4.1 Necessary Conditions at a BNE

Our goal is to investigate the properties of allocation rules that are implementable by assessments that consist a Perfect Bayesian Equilibrium. We first have to look at the restrictions imposed on p by requiring (σ, μ) to be a BNE of the game. At a BNE the buyer's strategy is a best response to the seller's strategy. The following Lemma establishes that if this is the case, then p is increasing in the buyer's valuation.

Lemma 1 *If σ_B is a best response to σ_S, then $p(\sigma, \mu)(v)$ is increasing in v.*

Proof. To see this consider an assessment (σ, μ) where the strategy of the seller σ_S is a best response to the strategy of the buyer σ_B. Let $U_{\sigma,\mu}(\sigma_B(v), v)$ denote the buyer's expected discounted payoff when his valuation is v given (σ, μ). It is given by

$$U_{\sigma,\mu}(\sigma_B(v), v) = p(v)v - x(v). \tag{1}$$

The buyer's payoff from adopting actions $\sigma_B(v')$ when his reservation value is v can be expressed as

$$U_{\sigma,\mu}(\sigma_B(v'), v) = p(v')v - x(v'), \tag{2}$$

where $\sigma_B(v')$ describes the actions specified by σ_B for the case that the buyer's valuation is v'. Analogously we can write $U_{\sigma,\mu}(\sigma_B(v'), v') = p(v')v' - x(v')$ and $U_{\sigma,\mu}(\sigma_B(v), v') = p(v)v' - x(v)$. Since σ_B is a best response to σ_S we have that

$$\begin{aligned} U_{\sigma,\mu}(\sigma_B(v), v) - U_{\sigma,\mu}(\sigma_B(v), v') &\geq 0 \\ U_{\sigma,\mu}(\sigma_B(v'), v') - U_{\sigma,\mu}(\sigma_B(v'), v) &\geq 0. \end{aligned} \tag{3}$$

From (3) we get that

$$
\begin{aligned}
& \left[U_{\sigma,\mu}(\sigma_B(v),v) - U_{\sigma,\mu}(\sigma_B(v'),v)\right] + \left[U_{\sigma,\mu}(\sigma_B(v'),v') - U_{\sigma,\mu}(\sigma_B(v),v')\right] \\
= \quad & \left(p(v') - p(v)\right)\left(v' - v\right) \geq 0.
\end{aligned}
$$

Hence if the buyer's strategy, σ_B, is a best response to σ_S, the allocation rule, p, will be increasing in v. ■

From standard arguments, (see for instance Myerson (1981)), it follows that $U_{\sigma,\mu}(\sigma_B(v),t)$ is increasing in t and hence differentiable almost everywhere. The fact that the buyer can always reject a mechanism offered by the seller, implies that the payoff of the buyer must be non-negative, that is

$$U_{\sigma,\mu}(\sigma_B(v),v) = p(v)v - x(v) \geq 0.$$

Expected discounted revenue given an assessment (σ,μ) that implements an allocation rule p can be written as

$$R = \int_a^b p(v)\left[v - \frac{(1-F(v))}{f(v)}\right] f(v)dv - U_{\sigma,\mu}(\sigma_B(a),a). \tag{4}$$

We will later establish that at a PBE we have $U_{\sigma,\mu}(\sigma_B(a),a) = 0$.

From the above analysis it follows that if the buyer's strategy is a best response to σ_S, p is an increasing function of v and expected discounted revenue for the seller given (σ,μ) is determined solely by p and the payoff that accrues to the buyer when his valuation is equal to the lowest possible.

We now proceed to investigate the structure that sequential rationality imposes on p. In order to do so, we need to study the seller's behavior at the beginning of the final period of the game.

4.2 The Seller's Problem at the Beginning of the Final Period of the Game

In a PBE the mechanism employed at $t=1$, M_1, must be an equilibrium of the continuation game that starts after a history h_S^1, where trade did not occur at $t=0$. That is, at $t=1$ M_1 must maximize expected revenue given posterior beliefs. In the case that the buyer's valuation is fully revealed after some history h_S^1, the seller's problem at $t=1$ is trivial. She names a price equal to the buyer's valuation and extracts all his surplus. In what follows we analyze the case where the seller is uncertain about the buyer's valuation at the beginning of period $t=1$. Since $t=1$ is the final period of the game the seller can, without loss of generality, choose M_1 among the class of direct revelation mechanisms, (DRM), that are incentive compatible, (IC), and individually rational, (IR). Consider a PBE (σ,μ) and a history along the equilibrium path h_S^1, where the buyer did not obtain the object at $t=0$.

The set of possible types at $t = 1$, given a history h_S^1, is denoted by $Y_{\sigma,\mu}(h_S^1)$. We will assume that it is measurable and that it has strictly positive measure. This is without loss for our analysis, since histories where $Y_{\sigma,\mu}(h_S^1)$ has measure zero do not matter from the ex-ante point of view. We will often write Y instead of $Y_{\sigma,\mu}(h_S^1)$. The PDF of v given Y is

$$f_1(v) = \begin{cases} \frac{f(v)}{\int_Y f(s)ds} \text{ if } v \in Y \\ 0 \text{ otherwise} \end{cases} . \tag{5}$$

The type space at $t = 1$, Y, is endogenous, since it depends on the history of the game. It may not be a closed and convex subset of the real line as it is usually assumed in the mechanism design literature under commitment.

A DRM consists of two mappings $r : Y \to [0, 1]$ and $z : Y \to \mathbb{R}_+$, where $r(v)$ specifies the probability of obtaining the object, if the buyer claims that his valuation is v, and $z(v)$ specifies the corresponding expected payment.

Consider a history h_S^1 where trade has not taken place up to $t = 1$. For every such history the mechanism that the seller will employ according to her equilibrium strategy, denoted by $M_1(h_S^1)$, must solve

$$\max_{(r,z)\in DRM} \int_Y z(v)f_1(v)dv, \tag{R$_1$}$$

subject to

$$r(v)v - z(v) \geq r(v')v - z(v') \text{ for all } v, v' \in Y \tag{IC}$$

and

$$r(v)v - z(v) \geq 0, \text{ for all } v \in Y. \tag{IR}$$

We will refer to the above maximization problem as **Program 1.**

Program 1 differs from a standard static problem in that the type space is not necessarily an interval. In what follows we show that requiring the mechanism to be feasible on the convex hull of the closure of Y is without any loss. We do that in steps. First we show that requiring IC and IR to hold for all types in the *closure* of Y, denoted by $[Y]$, does not change the solution of Program 1. In the second step we show that the same holds even if require the mechanism to satisfy IC and IR on the *convex hull* of Y, which we denote by $\bar{Y}$.

First consider a version of Program 1, where the mechanism that the seller employs, must satisfy IC and IR on $[Y]$. We will call this problem *Program 1b,* and it given by

$$\max_{r,z\in DRM} \int_{[Y]} z(v)f_1(v)dv$$

subject to

$$r(v)v - z(v) \geq r(v')v - z(v') \text{ for all } v, v' \in [Y] \tag{IC$_{cl}$}$$

and

$$r(v)v - z(v) \geq 0, \text{ for all } v \in [Y], \tag{IR$_{cl}$}$$

where $r : [Y] \to [0,1]$ and $z : [Y] \to \mathbb{R}_+$.

Since $f_1(v) = 0$ for $v \in [Y] \backslash Y$ the objective function in Program 1b is the *same* as in Program 1.

Let $R_1(M_1)$ denote the seller's expected revenue at the continuation game that starts at $t = 1$, when the seller employs M_1,

$$R_1(M_1) = \int_Y z(v) f_1(v) dv.$$

All proofs that are not included in the main text can be found in the appendix.

Lemma 2 *Let M_1 denote the solution of Program 1 and $\bar{M}_1$ denote the solution of Program 1b. Then*

$$R_1(M_1) = R_1(\bar{M}_1).$$

Given this result it follows that it is without loss to assume that the type space at $t = 1$ is a closed subset of the real line.

Second, consider a version of Program 1, where the mechanism that the seller employs, must satisfy IC and IR on the convex hull of Y.

Program 2:

$$\max_{r_E, z_E \in DRM} \int_{\bar{Y}} z_E(v) f_1(v) dv$$

subject to

$$r_E(v)v - z_E(v) \geq r_E(v')v - z_E(v') \text{ for all } v, v' \in \bar{Y} \tag{IC$_E$}$$

and

$$r_E(v)v - z_E(v) \geq 0, \text{ for all } v \in \bar{Y}, \tag{IR$_E$}$$

where $r_E : \bar{Y} \to [0,1]$ and $z_E : \bar{Y} \to \mathbb{R}_+$.

We will take Y to be closed. From Lemma 2 it follows that this is without loss of generality.

Proposition 1 [6]*Assume that Y is closed and let M_1^E denote the solution of Program 2. Then*

$$R_1(M_1) = R_1(M_1^E).$$

[6] The conjecture that such a result may be available arose from discussions with Kim-Sau Chung.

We now solve Program 2, and demonstrate that after any history, the seller will maximize revenue at the continuation game that starts at $t = 1$ by posting a price. In the analysis that follows we take the convex hull of the closure of Y to be $[a, b]$, but it can be any interval $[\underline{v}, \bar{v}]$, for some $\underline{v}, \bar{v} \in [a, b]$, with $\underline{v} < \bar{v}$. Following Myerson (1981), the seller's expected revenue at the beginning of $t = 1$ can be written as,

$$\int_a^b r(v)\left[v f_1(v) - [1 - F_1(v)]\right] dv - u_1(a), \tag{6}$$

where u_1 denotes the buyer's payoff at the continuation game that starts at period $t = 1$. Note that (6) can be equivalently written as $\int_a^b r(v)\left[v - \frac{[1-F_1(v)]}{f_1(v)}\right] f_1(v)dv$. In the problem at hand, f_1 is not necessarily strictly positive so this expression is not always well defined whereas the one given by (6) is.

Recall from the analysis in Myerson (1981), that the optimal mechanism should set $u_1(a) = 0$. Let

$$\phi_1(v) = v f_1(v) - [1 - F_1(v)],$$

then, (6) can be rewritten as

$$\max_{r \in \Im} \int_a^b r(v)\phi_1(v)dv, \tag{7}$$

where

$$\Im = \left\{ \begin{array}{c} r : [a, b] \to [0, 1] \text{ such that } r \text{ is} \\ \text{increasing} \end{array} \right\}. \tag{8}$$

The requirement that r be increasing follows from incentive compatibility, IC, and together with setting $u_1(a) = 0$ ensures IR. [7]

In the Proposition that follows, we characterize the optimal mechanism at the beginning of the final period of the game. Our objective is to choose a function $r \in \Im$ such that (7) is maximized. Ideally we would like to set r equal to 0 at the points where ϕ_1 is negative and equal to 1 at the points where ϕ_1 is positive. The constraint that r be monotonic does not allow this. In order to understand the nature of the solution given the constraint let us first examine the case that ϕ_1 is strictly increasing [8] in v. In this case the constraint of monotonicity is not binding. The optimal r is a step function that jumps from zero to one at the point where ϕ_1 starts to take positive value, that is at the solution of $\phi_1(v) = 0$. In the case where ϕ_1 is not monotonic the constraint is binding. Nonetheless, because (7) is linear in r it follows that the maximizer will be a step function that jumps from zero to one. The point where the optimal r jumps from zero to one is equal to the smallest v with the property that the area under ϕ_1 from v to any point greater of v is positive. This sketch is formalized in the proof of the Proposition that follows.

[7] For more details see Myerson (1981).

[8] This corresponds to the regular case in Myerson (1981).

Proposition 2 [9] *The mechanism given by*

$$r(v) = 1 \text{ if } v \geq z_1 \quad \text{and} \quad z(v) = z_1 \text{ if } v \geq z_1$$
$$= 0 \text{ if } v < z_1 \qquad\qquad = 0 \text{ if } v < z_1 \tag{9}$$

where

$$z_1 \equiv \inf \left\{ v \in [a,b] \text{ such that } \int_v^{\tilde{v}} \phi_1(s)ds \geq 0, \text{ for all } \tilde{v} \in [v,b] \right\}, \tag{10}$$

maximizes the seller's expected revenue in the continuation game that starts at $t = 1$.

Proposition 2 shows that the maximizer is one of the extreme points of $\Im$, and describes a way to find it. It states that the seller at $t = 1$ will maximize expected revenue from that point on by posting a price equal to z_1 given by (10). This price will depend of course on the history h_S^1. Since $t = 1$ is the last period of the game, the seller's problem is the same as in the case of commitment. We look at the one buyer case and assume that his type belongs in a measurable set. The approach developed in this section can be useful in many other asymmetric information environments where the agent's type space is complicated.

Our next result states a few properties of the price that the seller will post at $t = 1$. Let $z_1(Y)$ denote the solution of (10) when $f_1(v) = \begin{cases} \frac{f(v)}{\int_Y f(s)ds} & \text{if } v \in Y \\ 0 & \text{otherwise} \end{cases}$.

Lemma 3 *(i) If* $Y \subset \hat{Y}$ *then* $z_1(Y) \leq z_1(\hat{Y})$. *(ii) Let* $\Phi(v) = vf(v) - [1 - F(v)]$. *IF* $Y \subset \hat{Y} \subset [a,b]$, *then* $\int_{z_1(Y)}^{z_1(\hat{Y})} \Phi(s)ds < 0$. *(iii) If* $f_1(v) = \begin{cases} \frac{f(v)}{F(\bar{v})} & \text{if } v \in [a, \bar{v}] \\ 0 & \text{otherwise} \end{cases}$ *for some* $\bar{v} \in [a,b]$, *then* z_1 *is continuous and increasing in* $\bar{v}$.

This is an auxiliary result that will be used in the characterization of the revenue maximizing mechanism.

4.3 Necessary Conditions at a PBE

In this section we provide necessary conditions that an allocation rule satisfies if it is implemented by an assessment that is a PBE of the game. In the analysis that follows we assume that the buyer employs pure strategies. This is not an important restriction since there is a continuum of types. We provide only *necessary* conditions that an allocation satisfies if it is $PBE-$implementable.

Consider an assessment (σ, μ) and let (r, z) denote the contract chosen at $t = 0$ by the buyer when his valuation is equal to its lowest possible value, which is a. Moreover, let Y_a denote the set of types of the buyer that choose the same contract at $t = 0$ as type a. Suppose that $\bar{v}$ is the largest type on the closure of Y_a. Then Y_a is a subset of $[a, \bar{v}]$. When

[9] I thank Phil Reny for suggesting parts of the proof of Proposition 2.

the buyer chooses (r,z) the seller's belief is given by the conditional distribution of v given Y_a that is $f_1(v) = \begin{cases} \frac{f(v)}{\int_{Y_a} f(s)ds} \text{ if } v \in Y_a \\ 0 \text{ otherwise} \end{cases}$.

Proposition 3 *If an allocation rule is implemented at a PBE then it belongs in $\mathcal{P}^{PBE}$, where*

$$\mathcal{P}^{PBE} \equiv \left\{ \begin{array}{c} p : [a,b] \to [0,1], \textit{ increasing such that} \\ p(v) = r \textit{ for } v \in [a, z_1(Y_a)), \\ p(v) = r + (1-r)\delta \textit{ for } v \in [z_1(Y_a), \bar{v}) \\ r + (1-r)\delta \leq p(v) \leq 1 \textit{ for } v \in [\bar{v}, b] \\ \textit{for some } Y_a \subset V;\ r \in [0,1] \textit{ and } z_1(Y_a) \textit{ given} \\ \textit{by (10)} \end{array} \right\}.$$

We proceed to show that at a PBE expected discounted payoff of the buyer when his valuation is equal to its lowest possible value, is zero.

Corollary 1 *At a PBE it holds that $U_{\sigma,\mu}(\sigma_B(a), a) = 0$.*

4.4 The Revenue Maximizing PBE

We are looking for the maximum expected revenue that the seller can achieve at a PBE. In other words, we are searching for an allocation rule such that

$$p \in \arg \max_{p \in \mathcal{P}^{PBE}} R(p), \text{ where}$$

$$R(p) = \int_V p(v) \left[v - \frac{(1-F(v))}{f(v)} \right] f(v) dv. \tag{11}$$

A Benchmark: The Commitment Case

As a by-product of our analysis we derive the allocation rule that maximizes (11) among all increasing allocation rules.[10] It is given by

$$p^C(v) = \begin{cases} 1 \text{ if } v > z^C \\ 0 \text{ if } v \leq z^C \end{cases} \tag{12}$$

where $z^C$is given by

$$z^C \equiv \inf \left\{ v \in [a,b] \text{ such that } \int_v^{\tilde{v}} \Phi(s) ds > 0, \text{ for all } \tilde{v} \in [v,b] \right\}, \tag{13}$$

and

$$\Phi(v) = v f(v) - (1 - F(v)).$$

[10] This result is isomorphic to the one described in Proposition 2

There is a strategy profile that is a BNE of the game and it implements p^C. The seller posts a price z^Cat $t = 0$ and at $t = 1$; the buyer pays z^Cat $t = 0$ if $v > z^C$; otherwise he rejects the seller's offer. Given the strategy of the seller, the buyer's strategy is a best response. This is the BNE that yields the highest possible revenue for the seller.

Returning to our original problem, note first that p^C, the allocation rule that maximizes expected revenue among all $BNE-$implementable ones, is not feasible under non-commitment since $p^C \notin \mathcal{P}^{PBE}$.

We search among functions p that are elements of $\mathcal{P}^{PBE}$. For our purposes all assessments that lead to the same p will be considered as equivalent, since they raise the same expected revenue for the seller. We start by looking at the subset of $\mathcal{P}^{PBE}$ that contains the allocation rules implemented by strategy profiles where the closure of Y_a is convex. This set of allocation rules is denoted by $\mathcal{P}$. We demonstrate that for every element in $\mathcal{P}^{PBE}$ there exists an element of $\mathcal{P}$ that generates the same revenue. For this reason we focus on $\mathcal{P}$.

Suppose that $[Y_a]$ is convex, say it is $[a, \bar{v}]$ for some $\bar{v} \in (a, b]$. Then we have that

$$f_1(v) = \begin{cases} \frac{f(v)}{F(\bar{v})} \text{ if } v \in [a, \bar{v}] \\ 0 \text{ otherwise} \end{cases}, \text{ (for some } \bar{v} \in (a, b]). \quad (14)$$

When we discuss allocation rules implemented by $PBE's$ where $[Y_a]$ is convex, we will write z_1 as a function of $\bar{v}$. The set that contains these allocation rules is defined by

Definition 2 $\mathcal{P} \equiv \left\{ \begin{array}{c} p : [a, b] \to [0, 1], \textit{ increasing such that} \\ p(v) = r \textit{ for } v \in [0, z_1(\bar{v})), \\ p(v) = r + (1 - r)\delta \textit{ for } v \in [z_1(\bar{v}), \bar{v}) \\ r + (1 - r)\delta \leq p(v) \leq 1 \textit{ for } v \in [\bar{v}, 1] \\ \textit{for some } \bar{v} \in [a, b]; r \in [0, 1], \textit{ and } z_1 \textit{ given} \\ \textit{by (10)} \end{array} \right\}.$

We now show that it is without any loss to consider only the allocation rules in $\mathcal{P}$. We establish this by showing that for each $p \in \mathcal{P}^{PBE}$ there exists $\hat{p} \in \mathcal{P}$ such that $R(\hat{p}) \geq R(p)$.

Proposition 4 *For each $p \in \mathcal{P}^{PBE}$ there exists $\hat{p} \in \mathcal{P}$ such that $R(\hat{p}) \geq R(p)$.*

Now we consider a subset of $\mathcal{P}$, denoted by $\mathcal{P}^*$. Any allocation rule in $\mathcal{P}^*$ can be implemented by an assessment with the following two characteristics. First, M_0 contains two contracts; one that assigns the object with probability less than one, and a contract that assigns it with probability 1. Second, the buyer's action at $t = 0$ separates types into two groups, namely for some $\bar{v} \in [a, b]$, types in $[a, \bar{v})$, ("low" types), choose the low probability contract, and types in $[\bar{v}, b]$, ("high" types), choose the one that assigns the object with probability 1.

Definition 3

$$\mathcal{P}^* \equiv \left\{ \begin{array}{c} p : [a,b] \to [0,1], \text{ increasing such that} \\ p(v) = r \text{ for } v \in [a, z_1(\bar{v})), \\ p(v) = r + (1-r)\delta \text{ for } v \in [z_1(\bar{v}), \hat{v}) \\ p(v) = 1 \text{ for } v \in [\hat{v}, b] \\ \text{for some } \bar{v} \in [a,b]\, ;\ r \in [0,1]\, ;\ z_1(\bar{v}) \text{ given} \\ \text{by (10); and } \hat{v} \geq \bar{v}. \end{array} \right\}.$$

Note that $\mathcal{P}^* \subset \mathcal{P}$. This can be seen by taking $\bar{v}$ in the definition of $\mathcal{P}^*$ to equal $\hat{v}$. We proceed to show that the maximum of R over $\mathcal{P}$ is equal to the maximum of R over $\mathcal{P}^*$. This result tells us that the seller can do no better than employing a mechanism in $t = 0$ that separates types in to two broad groups, high and low ones. Moreover, this mechanism should assign the object with probability 1 to high types. Maximizing over this set of allocation rules is a straightforward task since its elements can be implemented by assessments where M_0 contains 2 contracts. Because $\mathcal{P}^*$ may contain allocation rules that are implemented by assessments that are not $PBE's$, we need to verify that the maximizer is indeed implemented by an assessment that is a PBE. In the final step we show that the optimal one can be implemented by a PBE of the game where the seller posts a price in each period.

We start by verifying that the maximum of R over $\mathcal{P}^*$ and $\mathcal{P}$ indeed exists.

Lemma 4 *The maximum of R over $\mathcal{P}$ and over $\mathcal{P}^*$ exists.*

Now we turn to establish an important relationship of the set $\mathcal{P}$ and $\mathcal{P}^*$. Namely, we show that every element of $\mathcal{P}$ can be approximated arbitrarily closely by a convex combination of elements of $\mathcal{P}^*$; $\mathcal{P}$ is in the convex hull of $\mathcal{P}^*$.

Lemma 5 *Every element of $\mathcal{P}$ can be approximated arbitrarily closely, in the usual metric, by a convex combination of elements of $\mathcal{P}^*$.*

Lemma 5 establishes that the allocation rules in $\mathcal{P}^*$ are extreme points of the set $\mathcal{P}$. Since the seller's expected revenue can be expressed as a linear function of p, the optimal allocation rule will be an extreme point of $\mathcal{P}$. This intuition is formalized in the following result.

Proposition 5 *Consider a linear function $R : \mathcal{P} \to \mathbb{R}$. Suppose that there exists a set $\mathcal{P}^* \subset \mathcal{P}$, such that every element of $\mathcal{P}$ can be approximated by a convex combination of elements of $\mathcal{P}^*$. Furthermore, suppose that the maximum value of R over $\mathcal{P}$ and $\mathcal{P}^*$ exists. Then*

$$\max_{p \in \mathcal{P}} R(p) = \max_{p \in \mathcal{P}^*} R(p).$$

A consequence of Proposition 5 is that we can focus on the problem of maximizing expected discounted revenue over the set $\mathcal{P}^*$. We proceed to show that the maximizer of this problem can be implemented by an assessment that is a PBE of the game where the seller posts a price in each period.

Any allocation rule in $\mathcal{P}^*$ can be implemented by an assessment with the following two characteristics. First, the seller proposes at $t = 0$ $M_0 = \{(r, z), (1, z_0)\}$, for some $(r, z) \in [0, 1] \times \mathbb{R}_+$ and $z_0 \in \mathbb{R}_+$ and at $t = 1$ proposes $M_1 = \{(0, 0), (1, z_1)\}$, for $z_1 \leq z_1(\hat{v})$, where $z_1(\hat{v})$ is given by (10) and $\hat{v} = \frac{z_0 - z - (1-r)\delta z_1}{1 - r - (1-r)\delta}$. Second, given M_1 and M_0 as above, the buyer's strategy, along the path, is a best response at each node. Type $\hat{v}$ is indifferent between choosing $(1, z_0)$ at $t = 0$ and choosing (r, z) at $t = 0$ and $(1, z_1)$ at $t = 1$.

In the final step we establish that the revenue maximizing rule among the elements of $\mathcal{P}^*$ can be implemented by a PBE of the game that the seller posts a price in each period. We do so in two steps. First, we show that for each allocation rule in $p \in \mathcal{P}^*$, implemented by an assessment where $z_1 < z_1(\hat{v})$, there exists an allocation rule $\tilde{p} \in \mathcal{P}^*$ where $z_1 = z_1(\hat{v})$, (that is where the seller behaves optimally at $t = 1$), and it generates higher revenue for the seller. Second, we show that the optimal M_0 contains the exit option $(0, 0)$ and a contract that assigns the object with probability 1. These claims are established in the proof of the following Lemma.

Lemma 6 *Let p^* denote the solution of $\max_{p \in \mathcal{P}^*} R(p)$. Then p^* can be implemented by a PBE of the game where the seller posts a price in each period.*

Now we are ready to state and prove the main result of the paper.

Theorem 1 *Under non-commitment the seller maximizes expected revenue by posting a price in each period.*

Proof. In Lemma 4, we verified that the seller's maximization problem is well defined. From Lemma 5 we know that an element of $\mathcal{P}$ can be written as a convex combination of elements of $\mathcal{P}^*$. The result follows from Proposition 5 and Lemma 6. ■

5 Commitment and Non-Commitment: Revenue Comparisons

In this section we compare the expected revenue for the seller when she employs a revenue maximizing mechanism under commitment and under non-commitment. Given commitment the revenue maximizing institution is to post a price equal to z^C, (given by (13)), in each period. We have shown that when the seller behaves sequentially rationally the revenue maximizing mechanism is to post a price in each period. Let z_0 denote the price posted at $t = 0$ and z_1 the price posted at $t = 1$. This sequence of prices has to be sequentially rational. The seller can replicate the situation under non-commitment in the commitment case by posting z_0 at $t = 0$ and z_1 at $t = 1$, instead of posting z^C in each period. From this observation it follows that in general

$$R_C \geq R_{NC}(\delta),$$

where R_C denotes the highest revenue that the seller can achieve under commitment and R_{NC} the highest revenue under non-commitment.

When the buyer and the seller are very patient, (in this model the buyer and the seller have the same discount factor), the seller will find it beneficial to move all trade in the last period of the game. In the last period of the game she has commitment power. If $\delta = 1$ by shifting all trade at t=1 she obtains expected revenue equal to R_C, which is the best she can hope for. It follows that when $\delta = 1$ expected revenue under commitment and under non-commitment coincide.

On the other hand, for δ very small the value of the object at t=1 is almost zero to the buyer no matter what his valuation is, so there is not much surplus for the seller to extract. When the seller and the buyer are very impatient the situation is almost equivalent to the full commitment case. The seller posts at t=0 the revenue maximizing price as in the environment with commitment; therefore we get that $R_{NC}(0) = R_C$.

From the above observations it follows that for extreme values of the discount factor the seller can achieve the same expected revenue under commitment and under non-commitment. For intermediate values of the discount factor it holds that $R_{NC} < R_C$. To get some idea about the magnitude of the difference we present an example.

Example 3 *Assume that the buyer's valuation is uniformly distributed on the interval [0,1]. First note that the optimal mechanism under commitment is to post a price* $z^C = 0.5$ *in each period. The corresponding expected revenue is* $R_C = 0.25$. *Now let us look at the non-commitment case. Let* $\bar{v}$ *denote the valuation of the buyer who is indifferent between accepting* z_0 *at* $t = 0$ *and accepting* z_1 *at* $t = 1$. *It is given by* $\bar{v} = \frac{z_0 - \delta z_1}{1-\delta}$. *For the assumed prior we have that, if the buyer rejects the price offer at* $t = 0$, *then* $F_1(t) = \frac{t}{\bar{v}}$. *The price posted at t=1 is given by* $z_1 = \frac{\bar{v}}{2}$. *Substituting this expression of* z_1 *into* $\bar{v}$ *we get that* $z_0 = \bar{v}\,(1 - 0.5\delta)$. *Given the above relationship between* z_1, $\bar{v}$ *and* z_0 *the seller will pick*

$$z_0 \in \arg\max \left\{ \delta \int_{z_1}^{\bar{v}} z_1 f(s)ds + \int_{\bar{v}}^{1} z_0 f(s)ds \right\}$$
$$\textit{where } \bar{v} = \frac{z_0}{1 - 0.5\delta} \textit{ and } z_1 = \frac{z_0}{2(1 - 0.5\delta)}.$$

The maximizer is given by

$$z_0 = \frac{(1 - 0.5\delta)^2}{2 - 1.5\delta}.$$

The following table gives the solution for different values of the discount factor.

Discount Factor δ	**Price at t=0, z_0**	**Price at t=1, z_1**	$\bar{\mathbf{v}} = \frac{z_0}{1-0.5\delta}$	$\mathbf{R}_{NC}$
0.0001	0.49999	0.25001	0.50002	0.24999
0.3	0.46612	0.27419	0.54839	0.23306
0.4	0.45714	0.28571	0.57143	0.22857
0.45	0.45330	0.29245	0.58491	0.22665
0.5	0.45	0.3	0.6	0.225
0.7	0.44474	0.34211	0.68422	0.22237
0.9	0.46538	0.42308	0.84615	0.23269
0.9999	0.49995	0.4999	0.9998	0.24998
1	0.5	0.5	1	0.25

6 Concluding Remarks

This paper establishes that the revenue maximizing allocation mechanism in a two-period model under non-commitment is to post a price in each period. It also develops a procedure to derive the optimal mechanism under non-commitment in asymmetric information environments. This method does not rely on the revelation principle.

Previous work has assumed that the seller's strategy is to post a price and the problem of the seller is to find what price to post. We provide a reason for the seller's choice to post a price, even though she can use infinitely many other possible institutions: posted price selling is the optimal strategy in the sense that it maximizes the seller's revenue. We hope that the methodology developed in this paper will prove useful in deriving the optimal dynamic incentive schemes under non-commitment in other asymmetric information environments.

In the future we plan to study the problem in an infinite-horizon framework, which may be a more appropriate model to study mechanism design under non-commitment. This problem is involved with issues which require careful analysis beyond the scope of this paper.

7 Appendix

Proof of Lemma 2

The solution of Program 1 has to satisfy IC and IR for $v \in Y$, whereas the solution of Program 1b has to satisfy these constraints for all $v \in [Y]$. From this observation it follows that

$$R_1(M_1) \geq R_1(\bar{M}_1).$$

We will now establish that

$$R_1(M_1) \leq R_1(\bar{M}_1).$$

We will argue by contradiction. Suppose not, then

$$R_1(M_1) > R_1(\bar{M}_1). \tag{15}$$

In what follows we extend M_1 on the closure of Y appropriately and show that this extension satisfies IC_{cl} and IR_{cl}.

First notice that by the definition of closure for each $v \in [Y]\backslash Y$ there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$ in Y such that $v_n \to v$ as $n \to \infty$. There may exist more than one sequence $\{v_n\}_{n\in\mathbb{N}}$ that converges to v. Each one of these sequence determines a corresponding sequence $r(v_n)$ and $z(v_n)$ which may have different limits.[11] The value of the mechanism is defined to be equal to the smallest of these limiting values. These limits are well defined since M_1 is feasible, which implies that r and z are increasing in v. The extension of M_1 on $[Y]$, denoted by M_1^{cl}, is defined by

$$\begin{aligned} r_{cl}(v) &= \begin{cases} r(v) \text{ if } v \in Y \\ \min_{\{v_n \to v\}} [\lim_{n\to\infty} r(v_n)] \text{ if } v \in [Y]\backslash Y \end{cases}, \\ z_{cl}(v) &= \begin{cases} z(v) \text{ if } v \in Y \\ \min_{\{v_n \to v\}} [\lim_{n\to\infty} z(v_n)] \text{ if } v \in [Y]\backslash Y \end{cases}. \end{aligned} \tag{16}$$

Since for all $v \in [Y]\backslash Y$ $f_1(v) = 0$ and $r_{cl}(v) = r(v)$ for all $v \in Y$, it holds that

$$R_1(M_1^{cl}) = R_1(M_1). \tag{17}$$

From (15) and (17) we obtain that

$$R_1(M_1^{cl}) > R_1(\bar{M}_1). \tag{18}$$

We proceed to demonstrate that M_1^{cl} satisfies IC_{cl} and IR_{cl}.

Step 1: We first show that it satisfies *incentive compatibility,* that is

$$r_{cl}(v)v - z_{cl}(v) \geq r_{cl}(v')v - z_{cl}(v') \text{ for all } v, v' \in [Y]. \tag{19}$$

By the definition of M_1^{cl} (19) is satisfied for $v \in Y$ and $v' \in Y$. Now take $v \in Y$ and consider $v' \in [Y]\backslash Y$. Since $v' \in [Y]\backslash Y$, there exists at least one sequence of elements of Y that converges to v'. Consider the one used to define the extension of M_1 on the closure of Y in (16) and denote it $\{v'_n\}_{n\in\mathbb{N}}$. Because M_1 satisfies IC on Y, then by the definition of M_1^{cl}, we have that

$$r_{cl}(v)v - z_{cl}(v) \geq r_{cl}(v'_n)v - z_{cl}(v'_n) \text{ for all } v'_n \in Y \tag{20}$$

Taking the limit of (20) we obtain

$$r_{cl}(v)v - z_{cl}(v) \geq r_{cl}(v')v - z_{cl}(v'), \tag{21}$$

[11] These limits exist because these are bounded sequences.

hence $v \in Y$ does not have incentive to report $v' \in [Y] \backslash Y$.

We now show that (19) is satisfied for $v \in [Y] \backslash Y$. Since $v \in [Y] \backslash Y$, there exists at least one sequence of elements of Y that converges to v. Consider the one used to define the extension of M_1 on the closure of Y in (16) and denote it $\{v_n\}_{n \in \mathbb{N}}$. So far we have established that

$$r_{cl}(v_n)v_n - z_{cl}(v_n) \geq r_{cl}(v')v_n - z_{cl}(v') \text{ for all } v' \in [Y] \text{ and } v_n \in Y.$$

Using the definition of M_1^{cl} and by taking the limit as $n \to \infty$, we obtain

$$r_{cl}(v)v - z_{cl}(v) \geq r_{cl}(v')v - z_{cl}(v') \text{ for all } v' \in [Y].$$

Step 2: Now we demonstrate that M_1^{cl} is individually rational. Since M_1^{cl} is incentive compatible it suffices to demonstrate that IR_{cl} holds for the smallest element of $[Y]$ call if $\underline{v}$. To see this, suppose that M_1^{cl} satisfies individual rationality for $\underline{v}$, that is

$$r_{cl}(\underline{v})\underline{v} - z_{cl}(\underline{v}) \geq 0. \tag{22}$$

But (22) implies

$$r_{cl}(\underline{v})v - z_{cl}(\underline{v}) \geq 0, \text{ for } v \in Y \text{ such that } v \geq \underline{v}$$

which combined with the fact that M_1 is IC leads to

$$r_{cl}(v)v - z_{cl}(v) \geq r_{cl}(\underline{v})v - z_{cl}(\underline{v}) \geq 0, \text{ for } v \in Y \text{ such that } v \geq \underline{v}.$$

Hence given that M_1^{cl} satisfies IC_{cl} it suffices to demonstrate IR_{cl} for $\underline{v}$.

If $\underline{v} \in Y$ then IR_{cl} follows from the fact that M_1 is individually rational. Now let's consider the case where $\underline{v} \notin Y$. Since $\underline{v} \in [Y] \backslash Y$, there exists at least one sequence of elements of Y that converges to $\underline{v}$. Consider the one used to define the extension of M_1 on the closure of Y in (16) and denote it $\{\underline{v}_n\}_{n \in \mathbb{N}}$. Since M_1 satisfies IR_{cl} we have that for all $n \in \mathbb{N}$

$$r(\underline{v}_n)\underline{v}_n - z(\underline{v}_n) \geq 0. \tag{23}$$

By the definition of M_1^{cl} (23) implies that

$$r_{cl}(\underline{v}_n)\underline{v}_n - z_{cl}(\underline{v}_n) \geq 0.$$

Taking the limit as $n \to \infty$ we obtain

$$r_{cl}(\underline{v})\underline{v} - z_{cl}(\underline{v}) \geq 0. \tag{24}$$

From the above observations it follows that M_1^{cl} satisfies IC_{cl} and IR_{cl}. Moreover as demonstrated in (18) it raises higher revenue than $\bar{M}_1$. Contradiction. ■

The following Lemma will be used in the proof of Proposition 1 and in the proof of Proposition 3.

Lemma A1. Suppose that there exist v_1, v_2 on the boundary of Y such that $v \notin Y$ for all $v \in (v_1, v_2)$. Then at a PBE equilibrium it must hold that

$$r(v_2)v_2 - z(v_2) = r(v_1)v_2 - z(v_1). \tag{25}$$

Proof. [12]Consider a PBE (σ, μ). Let Y denote the type space at the beginning of $t = 1$ after a history h_1^S. Also, let M_1 denote the mechanism the seller employs at $t = 1$ given h_1^S at the PBE under consideration. From Lemma 2 it follows that it is without loss to require M_1 to be feasible for all types on the closure of Y (instead of being feasible only on Y).

Because M_1 is incentive compatible it must hold that

$$r(v_2)v_2 - z(v_2) \geq r(v_1)v_2 - z(v_1).$$

We now demonstrate that at a PBE the above inequality must hold with equality, that is

$$r(v_2)v_2 - z(v_2) = r(v_1)v_2 - z(v_1). \tag{26}$$

To see this, we argue by contradiction. Suppose that

$$r(v_2)v_2 - z(v_2) > r(v_1)v_2 - z(v_1)$$

and modify M_1 as follows. For all types $v \geq v_2$, $v \in Y$, increase the payment by the constant Δz, where Δz is such that

$$r(v_2)v_2 - z(v_2) - \Delta z = r(v_1)v_2 - z(v_1). \tag{27}$$

We now show that the resulting DRM, call it $\hat{M}_1$, satisfies IC and IR.

Step 1: $\hat{M}_1$ is incentive compatible.

Take $v \in Y$, such that $v \leq v_1$. Since M_1 is IC we have

$$r(v)v - z(v) \geq r(v')v - z(v'), \text{ for all } v' \in Y,$$

which by the definition of $\hat{M}_1$ implies

$$\hat{r}(v)v - \hat{z}(v) \geq \hat{r}(v')v - \hat{z}(v'), \text{ for all } v' \in Y \text{ such that } v' \leq v_1.$$

Since $\Delta z > 0$, it holds that

$$r(v)v - z(v) \geq r(v')v - z(v') - \Delta z, \text{ for all } v' \in Y$$

which, using the definition of $\hat{M}_1$ can be rewritten as

$$\hat{r}(v)v - \hat{z}(v) \geq \hat{r}(v')v - \hat{z}(v'), \text{ for all } v' \in Y \text{ such that } v' \geq v_2.$$

[12]The proof of this Lemma is quite simple. It is included for completeness.

So far we have shown that if the buyer's type v is less or equal to v_1, he does not have an incentive to misreport.

We now show that if $v \geq v_2$, type-v buyer does not find profitable to report $v' \neq v$. We consider $v = v_2$. Since M_1 is incentive compatible we have that

$$r(v_2)v_2 - z(v_2) \geq r(v')v_2 - z(v') \text{ for all } v' \in Y. \tag{28}$$

Subtracting Δz from both sides of (28) and using the definition of $\hat{M}_1$ we obtain that

$$\hat{r}(v_2)v_2 - \hat{z}(v_2) \geq \hat{r}(v')v_2 - \hat{z}(v') \text{ for all } v' \in Y \text{ such that } v' > v_2. \tag{29}$$

So far we have shown that type v_2 does not have incentive to report $v' > v_2$.

Now we will demonstrate that v_2 does not have incentive to report $v' \leq v_1$. Since M_1 is incentive compatible we have that

$$r(v_1)v_1 - z(v_1) \geq r(v')v_1 - z(v') \text{ for all } v' \in Y. \tag{30}$$

For $v' \in Y$ such that $v_1 \geq v'$ we have that $r(v_1) \geq r(v')$. This follows from the fact that M_1 is IC. Because $v_2 > v_1$, this observation together with (30) imply that

$$r(v_1)v_2 - z(v_1) \geq r(v')v_2 - z(v') \text{ for all } v' \in Y,\ v_1 \geq v'. \tag{31}$$

Combining (31) with (27) we obtain that

$$r(v_2)v_2 - z(v_2) - \Delta z \geq r(v')v_2 - z(v') \text{ for all } v' \in Y,\ v_1 \geq v',$$

which using the definition of $\hat{M}_1$ can be rewritten as

$$\hat{r}(v_2)v_2 - \hat{z}(v_2) \geq \hat{r}(v')v_2 - \hat{z}(v') \text{ for all } v' \in Y \text{ such that } v_1 \geq v'. \tag{32}$$

From (29) and (32) it follows that v_2 does not have incentive to misreport. Therefore IC is satisfied for v_2. It is straightforward to show it is also satisfied for $v \geq v_2$. We have therefore demonstrated that $\hat{M}_1$ satisfies IC.

Step 2 : We show that $\hat{M}_1$ satisfies IR.

For $v \leq v_1$ IR of $\hat{M}_1$ follows from IR of M_1. For $v \geq v_2$ it suffices to check IR for v_2 (for a justification see Step 2 in the Proof of Lemma 2).

Since M_1 satisfies IR we have that

$$r(v_1)v_1 - z(v_1) \geq 0.$$

It follows that since $v_2 > v_1$

$$r(v_1)v_2 - z(v_1) \geq r(v_1)v_1 - z(v_1) \geq 0,$$

which together with (27) and the definition of $\hat{M}_1$ imply

$$\hat{r}(v_2)(v_2) - \hat{z}(v_2) = r(v_2)(v_2) - \delta z(v_2) - \delta \Delta z \geq 0$$

Hence the direct revelation mechanism $\hat{M}_1$ is IC and IR. Moreover, it raises strictly higher revenue than M_1. The seller has a profitable deviation at $t = 1$ contradicting the fact that we are considering a PBE. Hence (26) indeed holds. ■

Proof of Proposition 1

Because the solution of Program 1 has to satisfy IC and IR for $v \in Y$, whereas the solution of Program 2 has to satisfy these constraints for all $v \in \bar{Y}$, it follows that

$$R_1(M_1) \geq R_1(M_1^E).$$

We will now establish that

$$R_1(M_1) \leq R_1(M_1^E). \tag{33}$$

by contradiction. Suppose not, then

$$R_1(M_1) > R_1(M_1^E). \tag{34}$$

Now consider the extension of r and z on $\bar{Y}$, denoted by $\tilde{r}_E$ and $\tilde{z}_E$. We will call this direct revelation mechanism $\tilde{M}_1^E$. It is defined as follows

$$\begin{aligned} \tilde{r}_E(v) &= \begin{cases} r(v) \text{ if } v \in Y \\ r(\hat{v}) \text{ if } v \in \bar{Y} \backslash Y \end{cases}, \ \tilde{z}_E(v) = \begin{cases} z(v) \text{ if } v \in Y \\ z(\hat{v}) \text{ if } v \in \bar{Y} \backslash Y \end{cases}. \\ \text{where } \hat{v} &= \sup\{v' \in Y : v' \leq v\}. \end{aligned} \tag{35}$$

Since $f_1(v) = 0$ for all $v \in \bar{Y} \backslash Y$ and $\tilde{r}_E(v) = r(v)$ for all $v \in Y$ we have

$$R_1(\tilde{M}_1^E) = R_1(M_1). \tag{36}$$

From (34) and (36) we obtain that

$$R_1(\tilde{M}_1^E) > R_1(M_1^E).$$

We proceed to demonstrate that $\tilde{M}_1^E$ is feasible.

Step 1: Here we establish that $\tilde{M}_1^E$ satisfies incentive compatibility.

Consider a $v \in Y,$ since M_1 is feasible we have

$$r(v)v - z(v) \geq r(v')v - z(v') \text{ for all } v' \in Y. \tag{37}$$

By the definition of $\tilde{M}_1^E$ it follows that for $v \in Y$, $\tilde{r}_E(v) = r(v)$ and $\tilde{z}_E(v) = z(v)$; for $v' \in Y$, $\tilde{r}_E(v') = r(v')$ and $\tilde{z}_E(v') = z(v')$; and for $v' \in \bar{Y} \backslash Y$ $\tilde{r}_E(v') = r(\hat{v})$ and $\tilde{z}_E(v') = z(\hat{v})$ (where $\hat{v}$ defined in (35)). So from (37) we obtain that

$$\tilde{r}_E(v)v - \tilde{z}_E(v) \geq \tilde{r}_E(v')v - \tilde{z}_E(v') \text{ for all } v' \in \bar{Y}.$$

It follows that types in Y do not have incentive to misreport when the seller employs $\tilde{M}_1^E$.

Now consider a $v \in \bar{Y} \backslash Y$. Since Y is closed, there exists an open interval around v that has an empty intersection with Y. Consider the largest such interval; call it (v_1, v_2). From the fact that M_1 is a solution to Program 1 and since[13] $v_1, v_2 \in Y$, but no element of (v_1, v_2) is in Y, we know from Lemma A1 that

$$r(v_2)v_2 - z(v_2) = r(v_1)v_2 - z(v_1). \tag{38}$$

Consider a $v \in \bar{Y} \backslash Y$ and, in particular, a $v \in (v_1, v_2)$.

We first show that v does not have incentive to report $v' \in (v_1, v_2)$. Notice that by the definition of the extension of M_1 we have that

$$\tilde{r}_E(v) = r(v_1) \text{ and } \tilde{z}_E(v) = z(v_1) \text{ for all } v \in (v_1, v_2).$$

Trivially

$$r(v_1)v - z(v_1) \geq r(v_1)v - z(v_1),$$

which implies that

$$\tilde{r}_E(v)v - \tilde{z}_E(v) \geq \tilde{r}_E(v')v - \tilde{z}_E(v'), \text{ for all } v' \in (v_1, v_2). \tag{39}$$

Second, we show that v does not have incentive to report $v' \leq v_1$. Since M_1 is feasible we have that

$$r(v_1)v_1 - z(v_1) \geq r(v')v_1 - z(v'), \text{ for all } v' \in Y$$

which can be rewritten as

$$\left[r(v_1) - r(v')\right] v_1 \geq z(v_1) - z(v'), \text{ for all } v' \in Y. \tag{40}$$

From the fact that M_1 is IC, we have that for $v_1 \geq v'$, it holds that $r(v_1) \geq r(v')$. Since $v \in (v_1, v_2)$, (and hence $v > v_1$), (40) implies that

$$r(v_1)v - z(v_1) \geq r(v')v - z(v'), \text{ for all } v' \in Y \text{ such that } v' \leq v_1,$$

which gives

$$\tilde{r}_E(v)v - \tilde{z}_E(v) \geq \tilde{r}_E(v')v - \tilde{z}_E(v'), \text{ for all } v' \in \bar{Y} \text{ such that } v' \leq v_1. \tag{41}$$

The fact that (41) holds for types $v' \in \bar{Y}$, (and not just types $v' \in Y$), follows from the fact that $\tilde{M}_1^E$ for $v' \in \bar{Y} \backslash Y$ is equal to M_1 at some $\hat{v} \leq v'$ in Y.

Finally we demonstrate that v does not have incentive to report $v' \geq v_2$. From the feasibility of M_1 we obtain that

$$r(v_2)v_2 - z(v_2) \geq r(v')v_2 - z(v'), \text{ for all } v' \in Y$$

[13] Because Y is closed.

which can be rewritten as

$$\left[r(v_2) - r(v')\right] v_2 \geq z(v_2) - z(v'), \text{ for all } v' \in Y.$$

Because M_1 is IC it follows that for $v' \geq v_2$ we have that $r(v_2) \leq r(v')$. Since $v < v_2$ we obtain that

$$r(v_2)v - z(v_2) \geq r(v')v - z(v'), \text{ for all } v' \in Y \text{ such that } v' \geq v_2. \tag{42}$$

Now from (38), the fact that $r(v_2) \geq r(v_1)$ and the fact that $v_2 > v$ we obtain that

$$r(v_1)v - z(v_1) \geq r(v_2)v - z(v_2),$$

which combined with (42) gives us that

$$r(v_1)v - z(v_1) \geq r(v')v - z(v') \text{ for all } v' \in Y \text{ such that } v' \geq v_2.$$

From the definition of $\tilde{M}_1^E$ we obtain that

$$\tilde{r}_E(v)v - \tilde{z}_E(v) \geq \tilde{r}_E(v')v - \tilde{z}_E(v'), \text{ for all } v' \in \bar{Y} \text{ such that } v' \geq v_2. \tag{43}$$

From (39), (41) and (43) we see that $\tilde{M}_1^E$ satisfies IC_E for $v \in (v_1, v_2)$. Similarly one can check that IC_E is satisfied for all $v \in \bar{Y} \backslash Y$.

It remains to check $\tilde{M}_1^E$ satisfies IR_E.

Step 2: In this step we show that $\tilde{M}_1^E$ is individually rational.

First observe that if the IC_E constraints are satisfied then IR_E constraints are equivalent to[14]

$$\tilde{r}_E(\underline{v})\underline{v} - \tilde{z}_E(\underline{v}) \geq 0, \tag{44}$$

where $\underline{v}$ is the smallest type in $\bar{Y}$.

Since $\bar{Y}$ is the convex hull of Y and $\underline{v}$ is the smallest element of $\bar{Y}$, it is also the smallest element of Y. Because $\tilde{M}_1$ is feasible we have that

$$r(\underline{v})v - z(\underline{v}) \geq 0. \tag{45}$$

It follows by the definition of $\tilde{M}_1^E$ that (45) implies (44). From the above arguments it follows that $\tilde{M}_1^E$ is feasible. Moreover it raises strictly higher revenue for the seller than M_1^E. Contradiction, therefore

$$R_1(M_1) = R_1(M_1^E).$$

■

Proof of Proposition 2

Step 1 We start by proving existence of the solution of the seller's problem at the beginning of $t = 1$. Recall that $R_1(M_1)$ denotes the seller's expected revenue at the beginning

[14] See the Proof of Lemma 2 for more details.

of $t = 1$. From Proposition 1 it follows that it is without loss of generality to replace the type space with the convex hull of its closure. When the type space is convex, we can write R_1 as a function of r, that is

$$R_1(r) \equiv \int_a^b r(v)\phi_1(v)dv,$$

where $\phi_1(v) = vf_1(v) - [1 - F_1(v)]$.

The seller seeks to solve

$$\max_{r \in \Im} \int_a^b r(v)\phi_1(v)dv,$$

where $\Im = \{r : [a,b] \to [0,1]$, increasing$\}$.

Step 1a. (Sequential Compactness) In order to show sequential compactness of $\Im$ we will refer to the following results.

Theorem (A1). A sequence r_n of functions from X to W converges to a function r in the topology of pointwise convergence[15] if and only if for each $s \in X(= [a,b]$ in our problem), the sequence $r_n(s)$ of points of $W(= [0,1]$ in our problem) converges to the point $r(s)$. (For a proof see Munkres "Topology: A first Course" page 281.)

Let $\{r_n\}$ be a sequence of elements of $\Im$. Then, from Helly's Selection Principle, (see Kolmogorov and Fomin p. 372), it follows that there exists $r \in \Im$ and a subsequence of $\{r_n\}$ that converges pointwise to r. From Theorem A1 it also follows that there exists $r \in \Im$ and a subsequence of $\{r_n\}$ that converges to r. Hence every sequence in $\Im$ has a convergent subsequence. It follows that $\Im$ is sequentially compact.[16]

Step 1b. (Continuity) We want to show that the objective function is continuous on $\Im$ in the topology of pointwise convergence. In order to accomplish this we will use Lebesque's Dominated Convergence Theorem.

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Definition 4 *(Topology of pointwise convergence.) Given a point x of [0,1] and an open set U of space [0,1] let*

$$S(x,U) = \left\{ p \mid p \in [0,1]^{[0,1]} \text{ and } p(x) \in U \right\}$$

The sets $S(x,U)$ are a subbasis for a topology on $[0,1]^{[0,1]}$ which is called the topology of pointwise convergence. The typical basis element about a function p consists of all functions g that are close to p at finitely many points.

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Definition 5 *(Sequential Compactness). A topological space X is said to be sequentially compact if every infinite sequence from X has a convergent subsequence.*

Theorem (Lebesque's Dominated Convergence Theorem). Let g be a measurable function over a measurable set E, and suppose that $\{h_n\}$ is a sequence of measurable functions on E such that

$$|h_n(s)| \leq g(s)$$

and for almost all $s \in E$ we have $h_n(s) \to h(s)$. Then

$$\int_E h = \lim \int_E h_n.$$

(For a proof see Royden (1962) p.76.)

Take $E = [a, b]$ which is a measurable set, and g is given by $g(s) = g \; \forall \; s \in [a, b]$, where

$$g = \sup_{s \in [a,b]} r(s) \left[s f_1(s) - (1 - F_1(s)) \right]$$

Note that g is measurable, since it is a constant function, and is an upper bound for every function

$$h(s) = r(s) \left[s f_1(s) - (1 - F_1(s)) \right].$$

Because f is strictly positive and continuous on $[a, b]$ it is bounded, and so is f_1 and hence $g < \infty$.[17] Observe that h is a measurable function. Take $h_n(s) = r_n(s) \left[s f_1(s) - (1 - F_1(s)) \right]$ and apply Lebesque's Dominated Convergence Theorem with g defined as above.

Step 1c. We now demonstrate that a bounded and continuous function over a sequentially compact set has a maximum. First note that $R_1(r)$ is bounded by 1. Let $\bar{R}_1 = \sup_{r \in \Im} R_1(r)$ and let r_n be a sequence in $\Im$ such that

$$R_1(r_n) \geq \bar{R}_1 - \frac{1}{n}, \; n \in \mathbb{N}.$$

Since $\Im$ is sequentially compact, every sequence has a convergent subsequence, therefore $\{r_n\}_{n \in \mathbb{N}}$, has a convergent subsequence, $\{r_{n_{(1)}}\}_{n_{(1)} \in \mathbb{N}}$, that converges to $\bar{r}$. Since R_1 is continuous at $\bar{r}$, we have that $R_1(\bar{r}) = \lim_{n_{(1)} \to \infty} R_1(r_{n_{(1)}}) = \bar{R}_1$. Hence the maximum exists. ■

Step 2. So far we have established that the maximization problem given by (7) has a maximum we will now proceed to show that the maximizer is of the form

$$r_z(s) = \begin{cases} 1 \text{ if } s \geq z \\ 0 \text{ if } s < z. \end{cases} \tag{46}$$

The objective function is linear in the choice variable so the maximizer will be an extreme point of the set of $\Im$. The set of extreme points of $\Im$ is

$$K = \cup_{z \in [a,b]} r_z$$

[17] Recall that f_1 is given by (5) for some $Y \in [a, b]$.

where r_z is defined in (46).

Every increasing, non-negative function G with $G(1) = 1$ can be written as a convex combination of functions as defined in (46)

$$G(s) = \int_0^1 r_v(s) dG(v).$$

Let r^* be a maximizer of the problem defined in (7). Let R_1^* denote the maximum value of the objective function. Then using the above representation and Fubini's theorem we have

$$\begin{aligned}\int_a^b r^*(s)\phi_1(s)ds &= \int_a^b \left\{ \int_0^1 r_z(s)dr^*(z) \right\} \phi_1(s)ds = \\ &= \int_0^1 \left\{ \int_a^b r_z(s)\phi_1(s)ds \right\} dr^*(z) = R_1^*.\end{aligned}$$

This is a convex combination of functions of the form given in (46). Hence one of these functions is a maximizer. ■

Step 3. Now we turn to show that the mechanism given by

$$\begin{aligned} r(v) &= 1 \text{ if } v \geq z_1 \\ &= 0 \text{ if } v < z_1 \end{aligned} \quad \text{and} \quad \begin{aligned} z(v) &= z_1 \text{ if } v \geq z_1 \\ &= 0 \text{ if } v < z_1 \end{aligned} \tag{47}$$

where

$$z_1 \equiv \inf \left\{ v \in [a,b] \text{ such that } \int_v^{\tilde{v}} \phi_1(s)ds \geq 0, \text{ for all } \tilde{v} \in [v,b] \right\}, \tag{48}$$

maximizes the seller's expected revenue in the continuation game that starts at $t = 1$.

First note that z_1 is well-defined because the set

$$\left\{ v \in [a,b] \text{ such that } \int_v^{\tilde{v}} \phi_1(s)ds \geq 0, \text{ for all } \tilde{v} \in [v,b] \right\}$$

is non-empty since it contains b. Suppose that z_1 does not characterize the optimal mechanism at $t = 1$. We some abuse of notation, let $R_1(z_1)$ denote the seller's revenue at the beginning of $t = 1$ given z_1. If z_1 is not optimal then there exists $\tilde{z}_1$ such that $R_1(\tilde{z}_1) > R_1(z_1)$

First, suppose that $\tilde{z}_1 < z_1$. Then by the definition of z_1, there exists a $v' \in [\tilde{z}_1, z_1]$[18], such that

$$\int_{\tilde{z}_1}^{v'} \phi_1(s)ds < 0. \tag{49}$$

In this case expected revenue at the continuation game that starts at $t = 1$ is given by

$$R_1(\tilde{z}_1) = \int_a^{\tilde{z}_1} 0\phi_1(s)ds + \int_{\tilde{z}_1}^{v'} \phi_1(s)ds + \int_{v'}^b \phi_1(s)ds.$$

[18] Actually, from the definition of z_1 it follows that there exists $v' \in [\tilde{z}_1, b]$ such that $\int_{\tilde{z}_1}^{v'} \phi(t)dt$. A moment's thought will reveal that we can take $v' \leq z_1$ without any loss.

From (49) it follows that

$$\begin{aligned} R_1(\tilde{z}_1) &< \int_a^{\tilde{z}_1} 0\phi_1(s)ds + \int_{\tilde{z}_1}^{v'} 0\phi_1(s)ds + \int_{v'}^{z_1} \phi_1(s)ds + \int_{z_1}^{b} \phi_1(s)ds \\ &= R(z_1), \end{aligned}$$

contradiction.

Now suppose that $\tilde{z}_1 > z_1$. Then,

$$R_1(\tilde{z}_1) = \int_a^{z_1} 0\phi_1(s)ds + \int_{z_1}^{\tilde{z}_1} 0\phi_1(s)ds + \int_{\tilde{z}_1}^{b} \phi_1(s)ds.$$

From the definition of z_1 it follows that $\int_{z_1}^{\tilde{z}_1} \phi_1(s)ds \geq 0,$ hence

$$R_1(\tilde{z}_1) \leq \int_a^{z_1} 0\phi_1(s)ds + \int_{z_1}^{\tilde{z}_1} \phi_1(s)ds + \int_{\tilde{z}_1}^{b} \phi_1(s)ds = R_1(z_1),$$

contradiction. ■

Proof of Lemma 3

(i) Let $\underline{v}$ denote the smallest type in Y and $\bar{v}$ the largest one. Recall that we designate by z_1 the price posted at t=1 which is given by

$$z_1 = \inf \left\{ v \in [\underline{v}, \bar{v}] \text{ such that } \int_v^{\tilde{v}} (sf_1(s) - [1 - F_1(s)])\, ds \geq 0, \text{ for all } \tilde{v} \in [v, \bar{v}] \right\},$$

where $f_1(s) = \frac{f(s)}{\int_Y f(t)dt}$. Note that f_1 is continuous since f is. Define

$$\psi(s, Y) = sf(s) + F(s) - \int_Y f(t)dt \tag{50}$$

and observe that

$$z_1 = \inf \left\{ v \in [\underline{v}, \bar{v}] \text{ such that } \int_v^{\tilde{v}} \frac{\psi(s, Y)}{\int_Y f(t)dt} ds \geq 0, \text{ for all } \tilde{v} \in [v, \bar{v}] \right\}.$$

Because Y has positive measure we have that $\int_Y f(t)dt > 0$. Hence z_1 can be equivalently defined as

$$z_1 = \inf \left\{ v \in [\underline{v}, \bar{v}] \text{ such that } \int_v^{\tilde{v}} \psi(s, Y) ds \geq 0, \text{ for all } \tilde{v} \in [v, \bar{v}] \right\}. \tag{51}$$

If $Y \subset \hat{Y}$ then

$$\int_Y f(t)dt \leq \int_{\hat{Y}} f(t)dt. \tag{52}$$

We want to show that $z_1(Y) \leq z_1(\hat{Y})$. We argue by contradiction. Suppose that $z_1(Y) > z_1(\hat{Y})$. By (50) and (52) it follows that for all $v \in Y$ it holds that $\psi_1(v, \hat{Y}) \leq \psi_1(v, Y)$. This implies that if

$$\int_{z_1(\hat{Y})}^{v} \psi_1(s, \hat{Y}) ds \geq 0, \text{ for all } v \in \left[z_1(\hat{Y}), \bar{v}\right] \tag{53}$$

then

$$\int_{z_1(\hat{Y})}^{v} \psi_1(v, Y) ds \geq 0 \text{ for all } v \in \left[z_1(\hat{Y}), \bar{v}\right]. \tag{54}$$

By the definition of $z_1(\hat{Y})$ (53) holds; hence (54) also holds. But (54) together with the supposition that $z_1(\hat{Y}) < z_1(Y)$, contradict the definition of $z_1(Y)$. Hence it must hold that $z_1(\hat{Y}) \geq z_1(Y)$.

When Y is of the form $[a, \bar{v}]$ for some $\bar{v} \in [a, b]$, then we can consider z_1 and ψ as functions of $\bar{v}$. The above result implies that z_1 is an increasing function of $\bar{v}$.

(ii) We would like to establish that if $Y \subset \hat{Y} \subset V$ then $\int_{z_1(Y)}^{z_1(\hat{Y})} \Phi(s) ds < 0$, where $\Phi(v) = vf(v) - [1 - F(v)]$. This will done be establishing the following claim.

Claim 1. For all $v \in \left[z_1(Y), z_1(\hat{Y})\right]$ there exists $\tilde{v} \in \left[v, z_1(\hat{Y})\right]$ such that $\int_v^{\tilde{v}} \Phi(s) ds < 0$.

We will establish this by contradiction. Suppose that there exists $v \in \left[z_1(Y), z_1(\hat{Y})\right]$ such that for all $\tilde{v} \in \left[v, z_1(\hat{Y})\right]$ it holds that $\int_v^{\tilde{v}} \Phi(s) ds \geq 0$. First observe that $\psi(s, V) = \Phi(s)$. Now because $Y \subset \hat{Y} \subset V$ then if

$$\int_v^{\tilde{v}} \Phi(s) ds \geq 0, \text{ for all } \tilde{v} \in \left[v, z_1(\hat{Y})\right]$$

it holds that

$$\int_v^{\tilde{v}} \psi(s, \hat{Y}) ds \geq 0, \text{ for all } \tilde{v} \in \left[v, z_1(\hat{Y})\right]. \tag{55}$$

Let $\hat{v}$ denote the largest element of $\hat{Y}$. From the definition of $z_1(\hat{Y})$ we know that

$$\int_{z_1(\hat{Y})}^{\tilde{v}} \psi(s, \hat{Y}) ds \geq 0 \text{ for all } \tilde{v} \in \left[z_1(\hat{Y}), \hat{v}\right],$$

which together with (55) implies that

$$\int_v^{\tilde{v}} \psi(s, \hat{Y}) ds \geq 0 \text{ for all } \tilde{v} \in [v, \hat{v}], \tag{56}$$

which contradicts the definition of $z_1(\hat{Y})$. ■

Now consider type $z_1(Y)$. From Claim 1 we know that there exists $v_1 \in \left[z_1(Y), z_1(\hat{Y})\right]$ such that $\int_{z_1(Y)}^{v_1} \Phi(s) ds < 0$. If $v_1 = z_1(\hat{Y})$ we are done, otherwise by Claim 1 we know that there exists $v_2 \in \left[v_1, z_1(\hat{Y})\right]$ such that $\int_{v_1}^{v_2} \Phi(s) ds < 0$. If $v_2 = z_1(\hat{Y})$ we are done; otherwise by Claim 1 we know that there exists $v_3 \in \left[v_2, z_1(\hat{Y})\right]$ such that $\int_{v_2}^{v_3} \Phi(s) ds <$

0. Consider the sequence $\{v_n\}_{n\in\mathbb{N}}$ constructed in the way suggested above. This is an increasing sequence and it is straightforward to show that its limit is equal to $z_1(\hat{Y})$. It follows that $\int_{z_1(Y)}^{z_1(\hat{Y})} \Phi(s)ds < 0$.

(iii) We now demonstrate that when $f_1(v) = \begin{cases} \frac{f(v)}{F(\bar{v})} \text{ if } v \in [a, \bar{v}] \\ 0 \text{ otherwise} \end{cases}$, for some $\bar{v} \in (a, b]$, then z_1 is a continuous function of $\bar{v}$. Take $v_n \to \bar{v}$. We want to establish that $z_1(v_n) \to z(\bar{v})$. Now, for each $n \in \mathbb{N}$, $z_1(v_n)$ satisfies,

$$z_1(v_n) = \inf\left\{v \in [a, v_n] \text{ such that } \int_v^{\tilde{v}} \psi(s, v_n)ds \geq 0, \text{ for all } \tilde{v} \in [v, v_n]\right\}, \tag{57}$$

where

$$\psi(s, v_n) = sf(s) + F(s) - F(v_n)$$

Because f is continuous, so is F. From this observation it follows that when $v_n \to \bar{v}$, $\psi(s, v_n) \to \psi(s, \bar{v})$. Moreover ψ is bounded since f is. By Lebesque's' Dominated Convergence Theorem, (see Royden (1962 p. 76), we have $\int_v^{\tilde{v}} \psi(s, \bar{v})ds = \lim_{n\to\infty} \int_v^{\tilde{v}} \psi(s, v_n)ds$, since ψ is bounded . Taking the limit as $n \to \infty$ of (57) we have that

$$z_1(\bar{v}) = \inf\left\{v \in [a, \bar{v}] \text{ such that } \int_v^{\tilde{v}} \psi(s, \bar{v})ds \geq 0, \text{ for all } \tilde{v} \in [v, \bar{v}]\right\}.$$

■

Proof of Proposition 3

Consider a PBE assessment (σ, μ) and let p denote the allocation rule implemented by it. At a PBE the buyer's strategy is a best response to the seller's strategy. From Lemma 1 it follows that if an allocation rule is implemented by an assessment that is a PBE of the game, then p is increasing in v.

Moreover at a PBE the seller's strategy must be a best response at the continuation game that starts at $t = 0$. Let (r, z) denote the contract chosen by type a at $t = 0$ and Y_a the set of types that choose the same contract as type a at $t = 0$ at the assessment under consideration. The largest element of the closure of Y_a is denoted by $\bar{v}$; Y_a is a subset of $[a, \bar{v}]$. We use z_1 to denote the price that is optimal after the history that the buyer chooses (r, z) at $t = 0$ and trade does not take place at $t = 0$. Since at a PBE the buyer's strategy is a best response at the continuation game that starts at $t = 1$, we have that for $v \in Y_a$ such that $v < z_1$ the buyer will reject z_1 and for $v \in Y_a$ such that $v \geq z_1$ the buyer will accept z_1. Let

$$\begin{aligned} v_1 &= \sup\{v \in Y_a \text{ s.t. } v \text{ rejects } z_1 \text{ at } t = 1\} \\ v_2 &= \inf\{v \in Y_a \text{ s.t. } v \text{ accepts } z_1 \text{ at } t = 1\}. \end{aligned}$$

From the ex-ante point of view it holds that $p(v) = r$ for $v \in Y_a$ such that $v \leq v_1$ and $p(v) = r + (1 - r)\delta$ for $v \in Y_a$ such that $v \geq v_2$. By the monotonicity of p it follows that

$$p(v) = r \text{ for } v \in V \text{ such that } v < v_1 \text{ and}$$

$$p(v) = r + (1 - r)\delta \text{ for } v \in V \text{ such that } v_2 < v \leq \bar{v}.$$

Type v_1 is on the boundary of Y_a. If v_1 is in Y_a then $p(v_1) = r$ and $x(v_1) = z$; if v_1 is not in Y_a then it must be indifferent between choosing (r, z) at $t = 0$ and rejecting the seller's offer at $t = 1$, and his actions specified by σ_B. Similar considerations hold for type v_2 hence

$$\begin{aligned} p(v_1)v_1 - x(v_1) &= rv_1 - z \text{ and} \\ p(v_2)v_2 - x(v_2) &= rv_2 - z + (1 - r)\delta(v_2 - z_1). \end{aligned} \tag{58}$$

Note that all types in (v_1, v_2) do not belong in Y_a, but v_1 and v_2 are on the boundary of Y_a.

Let $\hat{r} : Y_a \to [0, 1]$ and $\hat{z} : Y_a \to \mathbb{R}_+$ denote the DRM that the seller will employ at $t = 1$, after the history that the buyer chooses (r, z) at $t = 0$ and he does not obtain the object. From Lemma A1 we know that the DRM that the seller employs at $t = 1$ must be such that $\hat{r}(v_1)v_2 - \hat{z}(v_1) = \hat{r}(v_2)v_2 - \hat{z}(v_2)$, which in the case under consideration it results to

$$v_2 - z_1 = 0, \text{ or } z_1 = v_2. \tag{59}$$

If $v_1 = v_2$ then it follows that

$$\begin{aligned} p(v) &= r \text{ for } v \in [a, z_1) \\ p(v) &= r + (1 - r)\delta \text{ for } v \in (z_1, \bar{v}] \text{ and} \\ r + (1 - r)\delta &\leq p(v) \leq 1 \text{ for } v \in (\bar{v}, b]. \end{aligned}$$

Now let us look at the case where $v_1 \neq v_2$. Substituting (59) in (58) we obtain that

$$p(v_2)v_2 - x(v_2) = rv_2 - z. \tag{60}$$

We now demonstrate that $p(v) = r$ for all $v \in (v_1, v_2)$. We will argue by contradiction. Suppose that there exists $v \in (v_1, v_2)$ such that $p(v) \neq r$. Note that since we are looking at a PBE it must be the case that

$$\begin{aligned} p(v)v - x(v) &\geq rv - z \text{ or} \\ [p(v) - r]\, v &\geq x(v) - z. \end{aligned}$$

Since p is increasing we have that $p(v) \geq r$ and because $p(v) \neq r$ it holds that $p(v) > r$. From this observation and the fact that $v_2 > v$, (recall that $v \in (v_1, v_2)$), we have that

$$\begin{aligned} [p(v) - r]\, v_2 &\geq x(v) - z \text{ or} \\ p(v)v_2 - x(v) &> rv_2 - z \end{aligned}$$

or by (60)

$$p(v)v_2 - x(v) > p(v_2)v_2 - x(v_2),$$

which implies that v_2 can benefit by mimicking the behavior of v. Contradiction. Therefore $p(v) = r$ for all $v \in (v_1, v_2)$.

From the above observations it follows that if p is an allocation rule implemented by an assessment that is a PBE, it is an increasing mapping from V to $[0,1]$ such that

$$\begin{aligned} p(v) &= r \text{ for } v \in [a, z_1) \\ p(v) &= r + (1-r)\delta \text{ for } v \in [z_1, \bar{v}) \text{ and} \\ r + (1-r)\delta &\leq p(v) \leq 1 \text{ for } v \in [\bar{v}, b]. \end{aligned}$$

■

Proof of Corollary 1

We argue by contradiction. Suppose that $U_{\sigma,\mu}(\sigma_B(a), a) = p(a)a - x(a) > 0$, then the seller has a profitable deviation at $t = 0$. Namely, she can increase the expected payments of all contracts in M_0 by Δx such that $p(a)a - x(a) - \Delta x = 0$. ■

Proof of Proposition 4

We want to show that for each $p \in \mathcal{P}^{PBE}$ there exists $\hat{p} \in \mathcal{P}$ such that $R(\hat{p}) \geq R(p)$. The result will be established by showing that every allocation rule in $\mathcal{P}^{PBE}$ implemented by an assessment where $[Y_a]$ is not convex, can be also implemented by an assessment where $[Y_a]$ is convex.

Consider an allocation rule in $p \in \mathcal{P}^{PBE}$ implemented by an assessment (σ, μ) where $[Y_a]$ is not convex. Suppose that the convex hull of $[Y_a]$ is $[a, \bar{v}]$. This allocation rule is given by

$$\begin{array}{c} p : [a, b] \to [0,1], \text{ increasing such that} \\ p(v) = r \text{ for } v \in [a, z_1(Y_a)), \\ p(v) = r + (1-r)\delta \text{ for } v \in [z_1(Y_a), \bar{v}) \\ r + (1-r)\delta \leq p(v) \leq 1 \text{ for } v \in [\bar{v}, b] \end{array}.$$

From Lemma 3 it follows that because $Y_a \subset [a, \bar{v}]$, then $z_1(Y_a) \leq z_1(\bar{v})$. Again from Lemma 3 (i) it follows that if $f_1(v) = \begin{cases} \frac{f(v)}{F(\bar{v})} \text{ if } v \in [a, \bar{v}] \\ 0 \text{ otherwise} \end{cases}$, for some $\bar{v} \in (a, b]$, then z_1 is increasing and continuous in $\bar{v}$. From this observation there must exist $\hat{v} \in (a, \bar{v})$ with the property that if $[Y_a] = [a, \hat{v}]$ the z_1 given by 10 is such that $z_1(\hat{v}) = z_1(Y_a)$.

Now consider an assessment where the set of types that choose the same contract as a is convex and that it implements the following allocation rule

$$\begin{aligned} \hat{p}(v) &= r \text{ for } v \in [a, z_1(\hat{v})) \\ \hat{p}(v) &= r + (1-r)\delta \text{ for } v \in [z_1(\hat{v}), \bar{v}) \\ r + (1-r)\delta &\leq \hat{p}(v) \leq 1 \text{ for } v \in [\bar{v}, b]. \end{aligned}$$

Such an assessment exists and it is indeed very similar to the one that implements p, call it (σ, μ). We just need to change slightly the strategy of the buyer. The reason for this similarity is simple. Types that are in $[a, \bar{v}]$ but are not in Y_a are indifferent between choosing (r, z) at $t = 0$ and choosing some other contract specified by σ_B. We consider 2 cases.

Case 1. $z_1(\bar{v}) = z_1(Y_a)$.

In this case the strategy of the seller is as before and the strategy of the buyer changes as follows. At $t = 0$ types in $[a, \bar{v})$ choose (r, z) from which at $t = 1$ types in $[a, z_1(\bar{v}))$ reject $z_1(\bar{v})$ whereas types in $[z_1(\bar{v}), \bar{v}]$ accept. (In the original assessment only types in $Y_a \subset [a, \bar{v})$ choose (r, z) at $t = 0$.) Types in $[\bar{v}, b]$ behave as before. Clearly this assessment implements $\hat{p}(v) = p(v)$ for all $v \in [a, b]$ and moreover $[\hat{Y}_a] = [a, \bar{v}]$.

Case 2. $z_1(Y_a) < z_1(\bar{v})$.

If $z_1(Y_a) < z_1(\bar{v})$ it can be shown that there are types in $(z_1(Y_a), \bar{v})$ that do not belong in Y_a. Call this set of types $-Y_a$. By the monotonicity of p we know that for $v \in -Y_a$ it must hold that $p(v) = r + (1 - r)\delta$. Now consider the assessment where the strategy of the seller is as before and the strategy of the buyer has been modified as follows. At $t = 0$ types in $[a, \hat{v})$ choose (r, z) and at $t = 1$ types in $[a, z_1(\hat{v}))$ reject the seller's offer, whereas types in $[z_1(Y_a), \hat{v})$ accept. Types in $(\hat{v}, \bar{v})$ choose the actions chosen by the types in $-Y_a$ in the assessment that implements p, and types in $[\bar{v}, b]$ behave as before. This assessment implements $\hat{p}(v) = p(v)$ for all $v \in [a, b]$ and moreover $[\hat{Y}_a] = [a, \hat{v}]$ for some $\hat{v} \in [a, b]$.

From the above analysis it follows that $R(\hat{p}) \geq R(p)$. ■

Proof of Lemma 4

We will prove that the maximum of R over $\mathcal{P}$ exists. Using an identical procedure one can show that the maximum of R over $\mathcal{P}^*$ exists.

Continuity. Continuity of R follows from an identical argument as the one used in Step 1b, in the proof of Proposition 2.

In order to prove that the maximum exists it remains to demonstrate that $\mathcal{P}$ is sequentially compact in the topology of pointwise convergence.

Sequential Compactness. We will first show that every sequence $p_n \in \mathcal{P}$, $n \in \mathbb{N}$ has a subsequence that converges pointwise to $p \in \mathcal{P}$. Recall that p_n:$[a, b] \to [0, 1]$, increasing and of the form

$$\begin{array}{c} p_n(v) = r \text{ for } v \in [a, z_1(\bar{v}_n)), \\ p_n(v) = r + (1-r)\delta \text{ for } v \in [z_1(\bar{v}_n), \bar{v}_n) \\ r + (1-r)\delta \leq p_n(v) \leq 1 \text{ for } v \in [\bar{v}_n, b] \end{array} ,$$

for some $v_n \in [a, b]$ and z_1 given by (10) for $f_1(v) = \begin{cases} \frac{f(v)}{F(v_n)} & v \in [a, v_n] \\ 0 \text{ otherwise} \end{cases}$.

Let $w_1, w_2, ...$ denote the rational points of $[a, b]$. Since p_n is bounded, the sequence $\{p_n\}$ has a subsequence, $\{p_n^{(1)}\}$ that converges at point w_1. Since $\{p_n^{(1)}\}$ is also bounded, it has a subsequence $\{p_n^{(2)}\}$ converging at the point w_2 as well as the point w_1; $\{p_n^{(2)}\}$ contains a subsequence $\{p_n^{(3)}\}$ that converges at point w_3 as well as at point w_1 and w_2 and so on. The "diagonal sequence" $\{h_n\} = \{p_n^{(n)}\}$ will then converge to every rational point of $[a, b]$. The limit of this subsequence, p, is an increasing function from $[a,b]$ to $[0,1]$. Moreover $p(s) = r$ for all the rationals in $[a, z_1)$, and since $z_1(.)$ is continuous, we have that $p(s) = r + (1-r)\delta$ for all the rationals in $[z_1(\bar{v}), \bar{v})$. We complete the definition of p at the remaining points of $[a,b]$ by setting[19]

$$p(v) = \lim_{v \to w^-} p(w) \text{ if } v \text{ is irrational.}$$

The resulting function p is then the limit of $\{h_n\}$ at every continuity point of p, (see Kolmogorov and Fomin page 373). Since p is increasing it has at most countably many discontinuity points. Using the diagonal process we can find a subsequence of $\{h_n\}$ that converges to all the discontinuity points p, which implies that it converges pointwise everywhere to p on $[a,b]$.

From the above arguments it follows that $\{p_n\}_{n\in\mathbb{N}}$ has a subsequence that converges pointwise to p which is an increasing function, such that at z_1 its value jumps from r to $r + (1-r)\delta$ and at $\bar{v}$ its value is $p(\bar{v}) = r + (1-r)\delta$, in other words, $p : [a,b] \to [0,1]$ is increasing and such that

$$\begin{array}{c} p(v) = r \text{ for } v \in [a, z_1(\bar{v})), \\ p(v) = r + (1-r)\delta \text{ for } v \in [z_1(\bar{v}), \bar{v}) \\ r + (1-r)\delta \leq p(v) \leq 1 \text{ for } v \in [\bar{v}, b] \end{array}.$$

Therefore $p \in \mathcal{P}$.

From Theorem A1, (stated in the proof of Proposition 2), it follows that $\{p_n\}_{n\in\mathbb{N}}$ has a subsequence that converges to p. Hence every sequence in $\mathcal{P}$ has a convergent subsequence. Therefore, $\mathcal{P}$ is sequentially compact. As seen in the proof of Proposition 2, **Step 1c**, a bounded continuous function on a sequentially compact set has a maximum. ■

Proof of Lemma 5

We will use p and q to denote generic elements of $\mathcal{P}$ and $\mathcal{P}^*$ respectively.

Every measurable function can be approximated by a step function in the usual metric generated by the norm (see for instance Royden 1962). An element of $\mathcal{P}$, say p, can be therefore approximated by a step function g. We now show that every step function that is arbitrarily close to an element of $\mathcal{P}$, can be written as a convex combination of elements of $\mathcal{P}^*$.

Take a $p \in \mathcal{P}$,

$$\begin{array}{c} p(v) = r \text{ for } v \in [a, z_1(\bar{v})), \\ p(v) = r + (1-r)\delta \text{ for } v \in [z_1(\bar{v}), \bar{v}) \\ r + (1-r)\delta \leq p(v) \leq 1 \text{ for } v \in [\bar{v}, b] \end{array},$$

and a step function g, such that $|p - g| < \varepsilon$, $\varepsilon > 0$ arbitrarily small. Since the restriction of p on $[a, \bar{v})$ is a step function, we can take $p(t) = g(t)$, for $t \in [a, \bar{v})$. Suppose that in the interval $[\bar{v}, b]$, g has K steps. Then we can consider the division of $[\bar{v}, b]$, into K subintervals,

[19] The notation $v \to w^-$ means that v approaches w from below.

I_j, $j = 1, ..., K$. In each of these subintervals g takes a potentially different value g_j, where $r + (1-r)\delta \le g_j \le b$.

We now show that we can write g as a linear combination of L functions $q_1, ..., q_L \in \mathcal{P}^*$, that is to say

$$g = \sum_{i=1}^{L} \alpha_i q_i,\ \Sigma_{i=1}^{L} \alpha_i = 1.$$

All $q_i's$ have the following characteristics

$$\begin{aligned} q_i(v) &= r,\ v \in [a, z_1(\bar{v})), \\ q_i(v) &= r + (1-r)\delta,\ v \in [z_1(\bar{v}), \hat{v}), \\ q_i(v) &= 1, v \in [\hat{v}, b], \end{aligned}$$

where $b \ge \hat{v} \ge \bar{v}$.

The way to determine the coefficients α_i, is as follows. Suppose that for $v \in I_1$, $g_1 = g(v) = r + (1-r)\delta + \eta_1$. Then for $v \in I_1$, I have $g(v) = \Sigma_{i=2}^{L} \alpha_i q_i + \alpha_1 q_1$, where $q_i = r + (1-r)\delta$ for all $i \ne 1$ and $q_1 = 1$, $\alpha_1 = \frac{\eta_1}{1-r-(1-r)\delta}$, and of course $\Sigma_{i=1}^{L} \alpha_i = 1$. (Observe that since $q_1 = 1$ on I_1 it must be $q_1 = 1$ for $v \in I_j$, $j = 2, ..., K$.) Obviously, $\Sigma_{i=2}^{L} \alpha_i = 1 - \alpha_1 = \frac{1-r-(1-r)\delta-\eta_1}{1-r-(1-r)\delta}$. So for $v \in I_1$, we can write $g(v) = \Sigma_{i=2}^{L} \alpha_i q_i + \alpha_1 q_1 = (r + (1-r)\delta) \cdot \left(\frac{1-r-(1-r)\delta-\eta_1}{1-r-(1-r)\delta}\right) + 1 \cdot \left(\frac{\eta_1}{1-r-(1-r)\delta}\right) = r + (1-r)\delta + \eta_1$. Now, suppose that for $v \in I_2$, we have $g_2 = r + (1-r)\delta + \eta_1 + \eta_2$. Then for $v \in I_2$ we can write $g(v) = \Sigma_{i=3}^{L} \alpha_i q_i + \alpha_1 q_1 + \alpha_2 q_2$, where $q_i = r + (1-r)\delta$ for all $i \ne 1, 2$ and $q_1 = 1 = q_2$, $\alpha_1 = \frac{\eta_1}{1-r-(1-r)\delta}$, $\alpha_2 = \frac{\eta_2}{1-r-(1-r)\delta}$ and $\Sigma_{i=1}^{L} \alpha_i = 1$. To verify this note that $\Sigma_{i=1, i \ne 1,2}^{L} \alpha_i = 1 - \alpha_1 - \alpha_2 = \frac{1-r-(1-r)-\eta_1-\eta_2}{1-r-(1-r)}$. We therefore obtain, that for $v \in I_2$, $g(v) = \Sigma_{i=1, i \ne 1,2}^{L} \alpha_i q_i + \alpha_1 q_1 + \alpha_2 q_2$

$= (r + (1-r)\delta) \cdot \left(\frac{1-r-(1-r)\delta-\eta_1-\eta_2}{1-r-(1-r)\delta}\right) + 1 \cdot \left(\frac{\eta_1}{1-r-(1-r)\delta}\right) + 1 \cdot \left(\frac{\eta_2}{1-r-(1-r)\delta}\right) = r + (1-r)\delta + \eta_1 + \eta_2$.

Continuing in a similar manner we can determine all the $\alpha_i' s$. It follows that any step function that is arbitrarily close to an element of $\mathcal{P}$, can be written as a convex combination of elements of $\mathcal{P}^*$. Therefore for each $p \in \mathcal{P}$ there exist a g, where g is a convex combination of elements of $\mathcal{P}^*$, such that $|p - g| < \varepsilon$. ■

Proof of Proposition 5

First note that since $\mathcal{P}^* \subset \mathcal{P}$, then

$$\max_{\mathcal{P}} R \ge \max_{\mathcal{P}^*} R.$$

It is given, that every element of $\mathcal{P}$ can be arbitrarily closely approximated by a convex combination of elements of $\mathcal{P}^*$. We will use p and q to denote generic elements of $\mathcal{P}$ and $\mathcal{P}^*$ respectively, and g to denote convex combinations of elements of $\mathcal{P}^*$. Suppose that $\bar{p} \in \mathcal{P}$ is a maximizer of R, and consider a sequence $\{g_n\}_{n \in \mathbb{N}}$ such that $g_n \to \bar{p}$. This implies that for all $\varepsilon > 0$, there exists g_n such that $|R(g_n) - R(\bar{p})| < \varepsilon$, for n large enough. From this we get that, for n large enough either $R(g_n) > R(\bar{p}) - \varepsilon$ or $R(\bar{p}) \geq R(g_n) - \varepsilon$.

Fix an n large enough. Since g_n is a convex combination of elements of $\mathcal{P}^*$, we can rewrite each element of this sequence as $g_n = \Sigma_{i=1}^{L} \alpha_i^n q_i^n$, where $q_i^n \in \mathcal{P}^*$ and $\Sigma_{i=1}^{L} \alpha_i^n = 1$. Then, because R is linear, we can write

$$R(g_n) = \Sigma_{i=1}^{L} \alpha_i^n R(q_i^n).$$

Now suppose that $R(q_i^n) < R(g_n)$ for all $i = 1, ..., L$. Then we have that $R(g_n) = \Sigma_{i=1}^{L} \alpha_i^n R(q_i^n) < R(g_n)$, but this is impossible. Hence there must exist i such that $R(q_i^n) \geq R(g_n)$. Now

$$\max_{\mathcal{P}^*} R(p) \geq R(q_i^n) \geq R(g_n),$$

where the first inequality follows from the fact that $q_i^n \in \mathcal{P}^*$. If $R(\bar{p}) > R(g_n)$ then

$$\max_{\mathcal{P}^*} R(p) \geq R(q_i^n) \geq R(g_n) > R(\bar{p}) - \varepsilon, \text{ for all } \varepsilon > 0.$$

Taking the limit as $\varepsilon \to 0$, we get that

$$\max_{\mathcal{P}^*} R(p) = R(\bar{g}) = \max_{\mathcal{P}} R(p).$$

If $R(g_n) \geq R(\bar{p})$, then from the fact that $q_i^n \in \mathcal{P}^*$ and $\mathcal{P}^* \subset \mathcal{P}$, we have

$$R(\bar{p}) \geq \max_{\mathcal{P}^*} R(p) \geq R(q_i^n) \geq R(g_n) \geq R(\bar{p}),$$

which again implies that all inequalities hold with equality. We therefore get

$$\max_{\mathcal{P}^*} R(p) = R(\bar{p}) = \max_{\mathcal{P}} R(p).$$

■

Proof of Lemma 6

Our objective is to demonstrate that the solution to the problem $\max_{p \in \mathcal{P}^*} R(p)$ can be implemented by an assessment that is a PBE of the game where the seller posts a price in each period.

Recall that any allocation rule in $\mathcal{P}^*$ can be implemented by an assessment with the following two characteristics.

C.1 First, the seller proposes at $t = 0$ $M_0 = \{(r, z), (1, z_0)\}$, for some $(r, z) \in [0, 1] \times \mathbb{R}_+$ and $z_0 \in \mathbb{R}_+$; and at $t = 1$ $M_1 = \{(0, 0), (1, z_1)\}$, for $z_1 \leq z_1(\hat{v})$, where $z_1(\hat{v})$ is given by (10) for $f_1(v) = \begin{cases} \frac{f(v)}{F(\hat{v})} \text{ if } v \in [a, \hat{v}] \\ 0 \text{ otherwise} \end{cases}$; and where

$$\hat{v} = \frac{z_0 - z - (1-r)\delta z_1}{1 - r - (1-r)\delta}. \tag{61}$$

C.2 Second, given M_1 and M_0 as above, the buyer's strategy, along the path, is a best response at each node. Type $\hat{v}$ is indifferent between choosing $(1, z_0)$ at $t = 0$ and choosing (r, z) at $t = 0$ and $(1, z_1)$ at $t = 1$.

We now establish that the revenue maximizing rule among the elements of $\mathcal{P}^*$ can be implemented by a PBE of the game that the seller posts a price in each period. We do so in two steps. First we show that for each allocation rule in $p \in \mathcal{P}^*$ implemented by an assessment where $z_1 < z_1(\hat{v})$ there exists an allocation rule $\tilde{p} \in \mathcal{P}^*$ where $z_1 = z_1(\hat{v})$, (that is the seller behaves optimally at $t = 1$), and it generates higher revenue for the seller. Second we show that at the optimum the seller posts a price in each period.

Step 1. Consider an allocation rule $p \in \mathcal{P}^*$ implemented by an assessment with the characteristics described in C 1 and C 2, where $z_1 < z_1(\hat{v})$. The expected discounted revenue for the seller is given by

$$\begin{aligned} R(p) &= \int_a^b p(s)\Phi(s)ds \\ &= \int_a^{z_1} r\Phi(s)ds + \int_{z_1}^{\hat{v}} (r + (1-r)\delta)\,\Phi(s)ds + \int_{\hat{v}}^b \Phi(s)ds. \end{aligned} \tag{62}$$

Now consider an assessment with $M_0 = \{(r, z), (1, \tilde{z}_0)\}$; $M_1 = \{(0, 0), (1, \tilde{z}_1)\}$, where $\tilde{z}_1 = z_1(\hat{v})$ and $\hat{v} = \frac{\tilde{z}_0 - z - (1-r)\delta\tilde{z}_1}{1-r-(1-r)\delta}$. Given M_0 and M_1 for $v \in [a, \tilde{z}_1)$ the buyer chooses (r, z) at $t = 0$ and rejects $\tilde{z}_1$ at $t = 1$; for $v \in [\tilde{z}_1, \hat{v})$ the buyer chooses (r, z) at $t = 0$ and accepts $\tilde{z}_1$ at $t = 1$; finally for $v \in [\hat{v}, b]$ the buyer chooses $(1, \tilde{z}_0)$ at $t = 0$. This assessment implements $\tilde{p} \in \mathcal{P}^*$. The expected revenue for the seller is given by

$$R(\tilde{p}) = \int_a^{\tilde{z}_1} r\Phi(s)ds + \int_{\tilde{z}_1}^{\hat{v}} (r + (1-r)\delta)\,\Phi(s)ds + \int_{\hat{v}}^b \Phi(s)ds. \tag{63}$$

Now subtracting (62) from (63) we obtain

$$R(\tilde{p}) - R(p) = \int_{z_1}^{\tilde{z}_1} (r + (1-r)\delta)\,\Phi(s)ds < 0.$$

This follows from Lemma 3 (ii) which gives us $\int_{z_1}^{\tilde{z}_1} \Phi(s)ds < 0$.

Step 2. Now we will establish that at the optimum the seller posts a price in each period. Recall that an allocation rule in $\mathcal{P}^*$ can be implemented by an assessment that satisfies the properties stated in C.1 and C.2. Given such an assessment the seller's revenue can be rewritten as

$$R(p) = \int_a^{z_1} zf(s)ds + \int_{z_1}^{\hat{v}} (z + (1-r)\delta z_1)\,f(s)ds + \int_{\hat{v}}^b z_0 f(s)ds, \tag{64}$$

where

$$z_1 \text{ is given by (10) for } f_1(v) = \begin{cases} \frac{f(v)}{F(\hat{v})} \text{ if } v \in [a, \hat{v}] \\ 0 \text{ otherwise} \end{cases}$$
$$\text{and } \hat{v} \text{ by (61).} \tag{65}$$

First from Corollary 1 it follows that it must hold that $ra - z = 0$ which implies that $z = ra$. Note also that given $z = ra$, using (61), z_0 can be written as a function of $\hat{v}$ as follows $z_0 = (1-r)(1-\delta)\hat{v} + ra + (1-r)\delta z_1(\hat{v})$. Substituting (65) into R we get

$$\begin{aligned} R(r,\hat{v}) &= raF(z_1(\hat{v})) + [ra + (1-r)\delta z_1(\hat{v})]\,[F(\hat{v}) - F(z_1(\hat{v})] \\ &\quad + [1 - F(\hat{v})]\,[(1-r)(1-\delta)\hat{v} + ra + (1-r)\delta z_1(\hat{v})]\,. \end{aligned}$$

The choice of the optimal M_0 has been reduced to the choice of the optimal r and $\hat{v}$.

The FOC with respect to r can be simplified to

$$\begin{aligned} \frac{\partial R}{\partial r} &= a - \delta z_1(\hat{v})\,[F(\hat{v}) - F(z_1(\hat{v})] \\ &\quad - ((1-\delta)\hat{v} + \delta z_1(\hat{v}))\,(1 - F(\hat{v}))\,; \end{aligned}$$

depending on the parameters this can be positive or negative. If $\frac{\partial R}{\partial r} < 0$ set r as small as possible that is $r = 0$, which implies $z = 0$; if $\frac{\partial R}{\partial r} > 0$, (which may be possible if a is sufficiently large), set r as large as possible that is $r = 1$ which implies that $z = a$. When at the optimum, $r = 1$ then the seller posts a price at $t = 0$ equal to the lowest possible valuation of the buyer, that is $z_0 = a$. Trade takes place with probability 1 at $t = 0$. When $r = 0$ at the optimum, M_0 contains $(0,0)$ and $(1, z_0)$ and M_1 contains $(0,0)$ and $(1, z_1)$.

In both cases the seller maximizes revenue by posting a price in each period. In the case that $\frac{\partial R}{\partial r} < 0$ she posts a price equal to z_0 at $t = 0$ and equal to z_1 at $t = 1$. The optimal level of z_0 and z_1 is determined by the optimal $\hat{v}$, which depends on f. In the case that $\frac{\partial R}{\partial r} > 0$ the seller posts a price equal to a at $t = 0$. ■

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