Judicial precedent as a dynamic rationale for axiomatic bargaining theory

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Abstract

Axiomatic bargaining theory (e.g., Nash's theorem) is static. We attempt to provide a dynamic justification for the theory. Suppose a Judge or Arbitrator must allocate utility in an (infinite) sequence of two-person problems; at each date, the Judge is presented with a utility possibility set in $\mathbb{R}^2_+$. He/she must choose an allocation in the set, constrained only by Nash's axioms, in the sense that a penalty is paid if and only if a utility allocation is chosen at date T which is inconsistent, according to one of the axioms, with a utility allocation chosen at some earlier date. Penalties are discounted with t, and the Judge chooses any allocation, at a given date, that minimizes the penalty he/she pays at that date. Under what conditions will the Judge's chosen allocations converge to the Nash allocation over time? We answer this question for three canonical axiomatic bargaining solutions: Nash's, Kalai-Smorodinsky's, and the 'egalitarian' solution.

Keywords: axiomatic bargaining theory, judicial precedent, dynamic foundations, Nash's bargaining solution.

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1 Introduction

Axiomatic bargaining theory is timeless. In Nash's (1950) original conception, the apparatus is meant to model a bargaining problem between two individuals, each of whom initially possesses an endowment of objects, and von Neumann-Morgenstern (vNM) preferences over lotteries on the allocation of these objects to the two individuals. An impasse point is defined as the pair of utilities each receives if no trade takes place, that is, if no bargain is reached (here, particular vNM utility functions are employed). Nash quickly passes to a formulation of the problem in utility space, where a bargaining problem becomes a convex, compact, comprehensive utility possibilities set, containing the impasse point. He then imposes the axioms of individual rationality, Pareto efficiency, symmetry, independence, and symmetry, and proves that the only "solution" satisfying these axioms on an unrestricted domain of problems is the Nash solution — for any problem, the utility point which maximizes the product of the individual gains from the threat point.

We say the theory is timeless, because of the independence axiom. For this axiom requires consistency in bargaining behavior between pairs of problems. What kind of experience might lead the bargainers to respect the independence axiom? Presumably, if they bargained for a sufficiently long period of time, facing many different problems, they might come across a pair of problems related as the premise of the independence axiom requires: problem S is contained in problem Q (as utility possibilities sets), the bargainers faced problem Q last year and chose allocation $q \in Q$, and it so happens that $q \in S$. It is certainly reasonable, they reason, to agree upon q when facing S this year, because of something like Le Chatelier's principle. ("If we chose q when all those allocations in $Q \backslash S$ were available, we effectively had decided to restrict our bargaining to S last year anyway, so let's choose $q \in S$ again now.") But if this is the way that bargainers might "learn" that independence is an attractive rule, then Nash's theory seems quite unrealistic. For with an unrestricted domain of problems, how often will bargainers face two problems related as the premise of the independence axiom requires? Almost never. Indeed, this fact is often cited as an argument in favor of Nash's theorems: the independence axiom is mathematically weak because it insists on coherence of the solution only in rare pairs of cases. Therefore, the independence restriction seems like a minimal one.

Notice the same argument of timelessness does not apply to the scale invariance axiom, even though that axiom compares the behavior of the solution on pairs of problems, because that axiom is meant to model the idea that only von Neumann Morgenstern preferences count, not their particular representation as utility functions. While the independence axiom can be viewed as a behavioral axiom, the scale invariance axiom is an informational axiom.

The other axioms — individual rationality, symmetry, and Pareto — are also behavioral but not timeless in our sense. It is not a mystery why bargainers should learn to cooperate (Pareto), or that two bargainers with the same preferences (and the same strengths) and the same endowments should end up at a symmetric allocation. Likewise, individual rationality is an easily justified behavioral axiom. Thus, the critique we are proposing of Nash bargaining theory is that one of the behavioral axioms (independence) has no apparent justification via some kind of learning through history, in the presence of another axiom (unrestricted domain), which essentially precludes that learning could ever take place.

Axiomatic bargaining theory has two major applications: one to bargaining, and the other to distributive justice. Of course, Nash (1950) pioneered the first interpretation, and the second was pioneered by Thomson and Lensberg (1989), who showed that many of the classical bargaining solutions (Nash, Kalai-Smorodinsky, egalitarian) could be characterized by sets of axioms with ethical interpretations. See Roemer (1996) for a history of the subject in its two variants.

Our goal in this article is to replace the timelessness of axiomatic bargaining theory with a dynamic approach, in which decision makers learn from history. Indeed, there is, we think, an obvious judicial practice, which provides a way rendering the theory dynamic. Suppose a judge or a court or an arbitrator faces a number of cases over time. There is a constitution that prescribes what the judicial decision must be in certain clear and polar cases. But most cases do not fit the specifications of these constitutionally described cases. So judges rely on judicial precedent or case law: they look for a case in the past that is similar in important respects, or related, to the one at hand, and decide the present case in like manner. Thus judicial precedent is a procedure providing a link to the past that is similar to the links between problems that the independence axiom — and, indeed, from a formal viewpoint, the scale invariance axiom — impose.

Of course, there is a possibility that the case being considered at present time, i, has two precedent cases j and k each of which is related to i in some important way, but which were decided differently. In general, the judge cannot decide the present case in a way to satisfy both precedents, and we will represent this conflict in our formal model. (Real judges tend to decide which precedent fits the case at hand more closely; but we will not follow this tack. Arguments occur around the proximity of various precedents to the case at hand.)

Imagine, then, that there is a *domain* of "cases" D, which is some set of Nash-type bargaining problems (convex, compact, comprehensive sets in $\mathbb{R}^2_+$). Suppose that the domain is rich enough that there are pairs of cases related by the scale invariance axiom, and pairs of cases related by the independence axiom; there are also some symmetrical cases in D. At each date $t = 1, 2, 3, \ldots$ a case is drawn randomly by Nature, according to some probability distribution on D. This infinite sequence of cases is called a history. The judge must decide each case sequentially (here, how to choose a feasible utility allocation), and he is restricted to obey the Nash axioms. What does this mean? If the case is symmetric, he must choose a symmetric point in the case, or pay a penalty of one; for every case, he must choose a Pareto efficient point, or pay a penalty of one. If a case is related to a prior case in the history by the scale-invariance or independence axiom, and he does not choose the allocation in the present case which is consistent with his prior choice according to the salient axiom, he must pay a penalty of δ^t , if the prior case appeared t periods ago, where $0 < \delta < 1$ is a given discount factor. (Thus, paying a penalty of one if a Pareto efficient point is not chosen in the case at hand is just a special case of this rule, because $\delta^0 = 1$.) If a case comes up which is not symmetric and is not related to any prior case by scale invariance or independence, he can choose any Pareto efficient point with zero penalty. At each date, the judge must choose an allocation which minimizes his penalty. In general, at a given date, he may end up paying penalties with respect a number of cases in the past which are precedents, and so his penalty would be a sum of the form $\sum_{i \in P} \delta^t$, for some set of non-negative integers P.

Now suppose that we consider a domain D where Nash's theorem is true: that is, any solution $\varphi : D \to \mathbb{R}^2_+$ satisfying $\varphi(i) \in i$ for all $i \in D$ that satisfies the Nash axioms on D is, in fact, the Nash solution on D, denoted N. Call such a domain a Nash domain. (The simplest Nash domain consists of precisely one symmetric set. Any solution on this domain must obey the symmetry axiom and Pareto. Thus any solution obeying the axioms coincides with N on this domain.) Our question is this: When is it the case that a judge who plays by the above rules, and faces an infinite history of cases, will converge over time almost surely to prescribing the Nash solution to the cases he faces?

To be precise, consider a super-domain $\mathfrak{H}^D$ of all possible histories over a given Nash domain, D, endowed with the product probability measure induced on histories by the given probability measure on D. When would the judge almost surely converge to prescribing the Nash solution as time passes on histories in $\mathfrak{H}^D$? We prove, under some simple additional assumptions, that convergence to the Nash solution occurs almost surely for every set of histories $\mathfrak{H}^D$, where D is any Nash domain, if and only if $0 < \delta \leq 1/3$ — that is, if and only if history is discounted at a sufficiently high rate. (Recall that the discount factor δ and the discount rate r are related by the formula $\delta = 1/(1+r)$.) This is our dynamic justification of Nash's theorem.

We extend the result to somewhat more complicated penalty systems and to several other classical axiomatic bargaining theorems. The rest of the paper is structured as follows. Section 2 introduces the axiomatic framework. Sections 3–5 successively deal with the Nash solution, the Kalai-Smorodinsky solution, and the egalitarian solution.

2 Framework and axioms

A *domain* $D = \{i, j, k, \ldots\}$ contains *problems*, namely, subsets of $\mathbb{R}^2_+$ which are compact, convex, and comprehensive.[1] For simplicity we also restrict attention to sets having a non-empty intersection with $\mathbb{R}^2_{++}$. Let ∂i denote the *upper frontier* of i, i.e.[2]

$$\partial i = \{x \in i \mid \nexists y \in i,\ y \gg x\},$$

and let $\partial^* i$ denote the subset of Pareto-efficient points of i :

$$\partial^* i = \{x \in i \mid \nexists y \in i,\ y > x\}.$$

Let $I(i)$ denote the vector of *ideal points*, i.e.

$$I(i) = (\max\{x_1 \in \mathbb{R}_+ \mid \exists x_2,\ (x_1, x_2) \in i\}, \max\{x_2 \in \mathbb{R}_+ \mid \exists x_1,\ (x_1, x_2) \in i\}).$$

For any $\alpha \in \mathbb{R}^2_{++}$, a set j is an α*–rescaling* of i if

$$j = \left\{x \in \mathbb{R}^2_+ \mid \exists y \in i,\ x_1 = \alpha_1 y_1,\ x_2 = \alpha_2 y_2\right\},$$

A *solution* $\varphi : D \to \mathbb{R}^2_+$ is a mapping such that for all $i \in D$, $\varphi(i) \in i$. The following axioms appear in the landmark theorems by Nash (1950), Kalai and Smorodinsky (1975)

[1] A set i is comprehensive when for all $x \in i$, all $y \leq x$, one has $y \in i$.

[2] Vector inequalities are denoted $\geq$, $>$, $\gg$.

and Kalai (1977):

Weak Pareto (WP): $\forall i \in D$, $\varphi(i) \in \partial i$.

Symmetry (Sym): $\forall i \in D$, if i is symmetric, then $\varphi_1(i) = \varphi_2(i)$.

Scale Invariance (ScInv): $\forall i, j \in D$, if j is an α–rescaling of i for some $\alpha \in \mathbb{R}^2_{++}$, then

$$\varphi(j) = (\alpha_1 \varphi_1(i), \alpha_2 \varphi_2(i)) .$$

Nash Independence (Ind): $\forall i, j \in D$, if $i \subseteq j$ and $\varphi(j) \in i$, then $\varphi(i) = \varphi(j)$.

Monotonicity (Mon): $\forall i, j \in D$, if $i \subseteq j$, then $\varphi(i) \leq \varphi(j)$.

Individual Monotonicity (IMon): $\forall i, j \in D$, $p \in \{1, 2\}$, if $i \subseteq j$ and $I_p(i) = I_p(j)$, then $\varphi_{3-p}(i) \leq \varphi_{3-p}(j)$.

Consider a finite domain D and an infinite number of periods $t = 1, 2, \ldots$ A *history* H is a sequence of problems and chosen points

$$H = ((i_1, x_1), (i_2, x_2) ...)$$

such that at every period t, $x_t \in i_t$. At each t a random process picks $i_t \in D$. For any given $i \in D$, the probability that $i_t = i$ may depend on the previous part of the history $((i_1, x_1), ..., (i_{t-1}, x_{t-1}))$. The random process is said to be *regular* if it never ascribes a zero probability (or a probability converging to zero) to any given problem, i.e., if for every $i \in D$, there $\pi_i > 0$ such that for every $t \in \mathbb{N}$, and for every past history $((i_1, x_1), ..., (i_{t-1}, x_{t-1}))$, the probability that $i_t = i$ is at least π_i.

At each period t the Judge chooses $x_t \in i_t$. His objective at each period is to minimize the penalty for this period, which is the sum of penalties incurred for a violation of each axiom. The values of such penalties, for each of the axioms, are the following (the parameters a, b, c, d, g, h are exogenous positive numbers, and $0 < \delta < 1$):

- (WP) a if $x_t \notin \partial i_t$.
- (Sym) b if i_t is symmetric and $\varphi_1(i_t) \neq \varphi_2(i_t)$.
- (ScInv) $\delta^s c$ if i_t is a α–rescaling of i_{t-s} and $\varphi(i_t) \neq (\alpha_1\varphi_1(i_{t-s}), \alpha_2\varphi_2(i_{t-s}))$.
- (Ind) $\delta^s d$ if $i_t \subseteq i_{t-s}$, $\varphi(i_{t-s}) \in i_t$ and $\varphi(i_t) \neq \varphi(i_{t-s})$.
- (IMon) $\delta^s g$ if for some $p \in \{1, 2\}$, either $i_{t+1} \subseteq i_t$, $I_p(i_{t+1}) = I_p(i_t)$, and $\varphi_{3-p}(i_{t+1}) \nleq \varphi_{3-p}(i_t)$; or $i_t \subseteq i_{t+1}$, $I_p(i_{t+1}) = I_p(i_t)$, and $\varphi_{3-p}(i_t) \nleq \varphi_{3-p}(i_{t+1})$.
- (Mon) $\delta^s h$ if either $i_{t+1} \subseteq i_t$ and $\varphi(i_{t+1}) \nleq \varphi(i_t)$; or $i_t \subseteq i_{t+1}$ and $\varphi(i_t) \nleq \varphi(i_{t+1})$.

The axioms involving a reference to past problems imply a penalty discounted by a factor δ : the farther back in the past the reference problem is, the lower the penalty.

Given a domain D and a random process selecting problems we say that the Judge *converges almost surely* to the solution φ if with probability 1 there is a date T such that for all $t \geq T$, the Judge chooses $\varphi(i_t)$.

3 Nash

The Nash solution, denoted N, is defined by

$$N(i) = \{x \in i \mid \forall y \in i,\ x_1x_2 \geq y_1y_2\}.$$

Nash's theorem is said to hold on D if $N(.)$ is the only solution satisfying WP, Sym, ScInv and Ind on D.

Let $i \in D$. Condition (N) is said to hold for i if there exists a sequence $j_1, \ldots j_n$ such that $j_1 = i$, j_n is symmetric and for all $t = 1, \ldots, n-1$, either

(i) $j_t \subseteq j_{t+1}$and $N(j_{t+1}) \in j_t$; or

(ii) $\exists \alpha \in \mathbb{R}^2_{++}$, j_{t+1} is an α–rescaling of j_t.

Call such a sequence a *special chain beginning at* i.

Proposition 1 *Nash's theorem holds on D if and only if Condition (N) holds for all $i \in D$.*

Proof. If: Let φ be any solution on D satisfying Nash's axioms. Let $i \in D$. By Condition (N) there is a special chain $j_1, \ldots j_n$ beginning at i. By Sym, $\varphi(j_n) = N(j_n)$. One can now roll back along the special chain to i, and at each step $\varphi(j_k) = N(j_k)$ either by Ind (case (i)) or by ScInv (case (ii)). For $k = 1$, we have $\varphi(i) = N(i)$. It follows that $\varphi = N$ on D.

Only if: Let D_+ be the subset of D containing the problems i for which Condition (N) holds. We must show that $D_+ = D$ if the Nash theorem holds on D. Suppose that there exists a problem $k \in D \setminus D_+$. Note that k is not symmetric because symmetric problems are in D_+. Similarly, k cannot be related by ScInv or Ind to a problem $i \in D_+$ because otherwise there exists a special chain $k, i, ...$, implying $k \in D_+$.

Observe that for all $i \in D$, $N(i) \in \partial^* i$. Consider the possibility that $\partial^* k$ is a singleton $\{x^*\}$. This can happen only if $k = \{x \in \mathbb{R}^2_{++} \mid x \leq x^*\}$. Then the α–rescaling of k, with $\alpha = (1/x_1^*, 1/x_2^*)$, is symmetric, which is impossible as shown in the previous paragraph. Therefore $\partial^* k$ is not a singleton, implying that it contains a point $\hat{x}$ such that $\hat{x} \neq N(k)$. By convexity of k, for every point $x \in \partial^* k$ there is $a \in (0,1)$ such that $x_1^a x_2^{1-a} \geq y_1^a y_2^{1-a}$ for all $y \in k$. Let $\hat{a}$ be a number satisfying this property for $\hat{x}$.

Consider the asymmetric Nash solution $N_{\hat{a}}$ defined by

$$N_{\hat{a}}(i) = \left\{x \in i \mid \forall y \in i,\ x_1^{\hat{a}} x_2^{1-\hat{a}} \geq y_1^{\hat{a}} y_2^{1-\hat{a}}\right\}.$$

On D, $N_{\hat{a}}$ satisfies WP, ScInv and Ind but not Sym. Define a new solution φ as follows:

$$\varphi(i) = \begin{cases} N(i) \text{ if } i \in D_+ \\ N_{\hat{a}}(i) \text{ if } i \in D \setminus D_+. \end{cases}$$

We show that φ satisfies all the axioms of the Nash theorem on D. It satisfies WP because its two component solutions satisfy it on their respective subdomains. It satisfies Sym because all symmetric problems are in D_+ and N satisfies Sym. Finally, it satisfies ScInv and Ind because a problem in $D \setminus D_+$ cannot be related to a problem in D_+ by these axioms, while on each subdomain these two axioms are satisfied.

As $\varphi(k) \neq N(k)$, the Nash theorem fails on D. Conversely, if the Nash theorem holds on D, then $D \setminus D_+$ must be empty. ■

We can now study the convergence of the Judge's decisions toward the Nash solution. The following proposition states that with probability one the Judge's decisions will exactly coincide with the Nash solution within a finite number of periods.

Proposition 2 *Assume that the random process is regular and that Condition (N) holds for all $i \in D$. If*

$$0 < \delta \leq \frac{\min\{a, b, c, d\}}{\min\{a, b, c, d\} + c + d}, \tag{1}$$

then the Judge converges almost surely to the Nash solution.

Proof. 1. Enumerate the problems in D as $1, 2, \ldots, M$. For each problem i define the special chain beginning at i as $i, j_2(i), ..., j_{n(i)}(i)$. Consider the following sequence of problems:

$$j_{n(1)}(1), j_{n(1)-1}(1), ..., 1, j_{n(2)}(2), ..., 2, j_{n(3)}(3), ..., 3, ..., j_{n(M)}(M), ..., M.$$

At every period, the probability that this sequence will occur at the next period is, by the assumption that the random process is regular, at least

$$\pi_{j_{n(1)}(1)}\pi_{j_{n(1)-1}(1)}...\pi_1\pi_{j_{n(2)}(2)}...\pi_2...\pi_{j_{n(M)}(M)}...\pi_M > 0.$$

Therefore, with probability one, this sequence occurs at a finite date T.

If $N(j_{n(1)}(1))$ is not chosen, the penalty will be at least $\min\{a,b\}$, since either WP or Sym will be violated. If, however, $N(j_{n(1)}(1))$ is chosen, this will entail at most two violations with respect to all previous choices — namely, for any previous date, a violation of ScInv and/or Ind. So the worst penalty that can be incurred is $(c+d)\sum_{t=1}^{T-1}\delta^t$. As $\delta > 0$,

$$(c+d)\sum_{t=1}^{T-1}\delta^t < (c+d)\sum_{t=1}^{\infty}\delta^t = (c+d)\frac{\delta}{1-\delta}.$$

Since

$$\frac{\min\{a,b,c,d\}}{\min\{a,b,c,d\}+c+d} \le \frac{\min\{a,b\}}{\min\{a,b\}+c+d}$$

by (1) one has

$$\delta \le \frac{\min\{a,b\}}{\min\{a,b\}+c+d},$$

which is equivalent to

$$(c+d)\frac{\delta}{1-\delta} \le \min\{a,b\},$$

so the Judge will choose $N(j_{n(1)}(1))$. Indeed, this argument shows that any symmetric problem will be assigned the Nash point by the Judge, when it occurs.

2. Now consider the second element in the sequence. If Judge does not choose $N(j_{n(1)-1}(1))$, he violates either ScInv or Ind w.r.t. the previous date, so the penalty is at least $\delta\min\{c,d\}$. If he does choose $N(j_{n(1)-1}(1))$, he is penalized at most

$$(c+d)\sum_{t=2}^{T}\delta^t < (c+d)\sum_{t=2}^{\infty}\delta^t = (c+d)\frac{\delta^2}{1-\delta},$$

and by the same argument as in the previous step (substituting c, d for a, b),

$$(c+d)\frac{\delta^2}{1-\delta} \leq \delta \min\{c, d\},$$

which implies that the Judge chooses $N(j_{n(1)-1}(1))$. In this way we see that we have the Nash choice on the whole sequence.

3. Now let the element that occurs after this sequence be i. If the Judge does not choose $N(i)$, he violates two axioms with respect to the previous occurrence of i in the sequence — namely, ScInv and Ind. The penalty is therefore at least $(c+d)\delta^t$ for some $1 \leq t \leq \sum_{j=2}^{M} n(j) + 1$. (The worst case is when $i = 1$.) Let $Q = \sum_{j=2}^{M} n(j) + 1$. On the other hand if he chooses $N(i)$, he at most violates ScInv and Ind with respect to all problems preceding the sequence (from the beginning of the history till $Q+1$ periods before) and is therefore penalized by less than

$$(c+d)\sum_{t=Q+1}^{\infty} \delta^t = (c+d)\frac{\delta^{Q+1}}{1-\delta}.$$

So he will choose $N(i)$ as long as

$$(c+d)\frac{\delta^{Q+1}}{1-\delta} \leq (c+d)\delta^Q.$$

This is equivalent to

$$\frac{\delta}{1-\delta} \leq 1,$$

which is true for

$$\delta \leq \frac{\min\{a, b, c, d\}}{\min\{a, b, c, d\} + c + d} \leq 1/3 < 1/2.$$

4. Assume that the Judge has chosen the Nash point for S periods after the end of the sequence (in the previous step we have shown this to be true for $S = 1$). Let the element that occurs at $S+1$ be i. If the Judge does not choose $N(i)$, he violates at

least ScInv and Ind with respect to the previous occurrence of i in the sequence and the penalty is therefore at least $(c+d)\delta^t$ for some $S+1 \leq t \leq S+Q$. If he chooses $N(i)$, he at most violates ScInv and Ind with respect to all problems preceding the sequence (from the beginning of the history till $S+Q+1$ periods before) and is is penalized by less than

$$(c+d)\sum_{t=S+Q+1}^{\infty}\delta^t = (c+d)\frac{\delta^{S+Q+1}}{1-\delta}.$$

So he will choose $N(i)$ as long as

$$(c+d)\frac{\delta^{S+Q+1}}{1-\delta} \leq (c+d)\delta^{S+Q}.$$

This is true because it is equivalent to $\frac{\delta}{1-\delta} \leq 1$.

By induction he chooses Nash henceforth. ■

Although (1) is a sufficient but not a necessary condition for the result to hold, we now show that a slightly weaker condition, $0 < \delta \leq \frac{b}{b+c+d}$, is necessary.

Proposition 3 *Assume that the random process is regular. If* $\delta = 0$ *or* $\frac{b}{b+c+d} < \delta < 1$, *there exists a domain* D *such that Condition (N) holds for all* $i \in D$ *but convergence to the Nash solution does not occur almost surely.*

Proof. Let $D = \{i, j\}$, as described in Fig. 1. The problem j is symmetric.

1. Suppose $\delta = 0$. Then ScInv and Ind have no influence on the Judge because violating them entails no penalty. He chooses $N(j)$ to avoid violating Sym on j, but can choose any point on ∂i (a point out of ∂i would entail a penalty for violation of WP). This holds independently of the order of occurrence of these problems in the history under consideration.

2. Suppose $\frac{b}{b+c+d} < \delta < 1$. The first inequality is equivalent to

$$\frac{b}{c+d}\left(\frac{1-\delta}{\delta}\right) < 1.$$

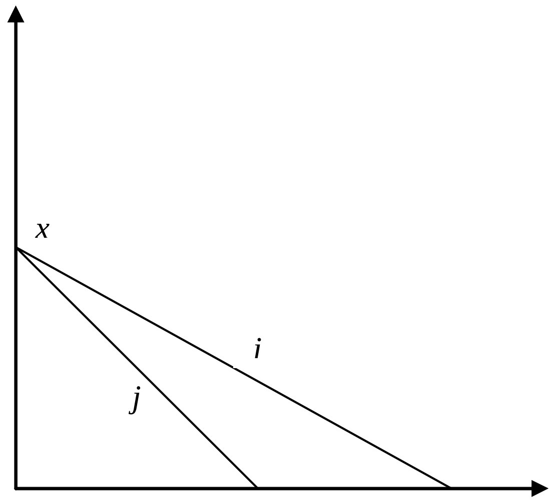

Figure 1: Example: the domain $D = \{i, j\}$

Let T be an integer satisfying

$$T > \frac{\ln\left(1 - \frac{b}{c+d}\left(\frac{1-\delta}{\delta}\right)\right)}{\ln\delta}.$$

There is a positive probability (at least π_i^T) that history starts with T occurrences of i. Suppose the Judge picks point x in i for $t = 1, ..., T$.

Let j occur at $t = T + 1$. If the Judge chooses $N(j)$ he violates ScInv and Ind w.r.t. the previous T periods and the penalty is $(c + d) \sum_{t=1}^{T} \delta^t$. If he chooses x he violates Sym and the penalty is b. If he chooses another point the penalty is either a or b, plus $(c + d) \sum_{t=1}^{T} \delta^t$. This last option is therefore dominated by $N(j)$. We have

$$\begin{aligned}(c+d)\sum_{t=1}^{T}\delta^t = (c+d)\delta\frac{1-\delta^T}{1-\delta} > b \Leftrightarrow \ln\left(1 - \frac{b}{c+d}\left(\frac{1-\delta}{\delta}\right)\right) > T\ln\delta \\ \Leftrightarrow T > \frac{\ln\left(1 - \frac{b}{c+d}\left(\frac{1-\delta}{\delta}\right)\right)}{\ln\delta},\end{aligned}$$

which is true by assumption. Therefore the judge picks x.

Consider a period $S > T + 1$ and assume that x has been chosen at all times before (we know this to be true for $S=T+2$). If i occurs, x is picked again without any penalty while any other point costs a penalty. If j occurs, picking x costs b while picking $N(j)$ costs $(c+d)\sum_{t=1}^{S-1}\delta^t > (c+d)\sum_{t=1}^{T}\delta^t$. So, again x is chosen.

By induction, at no period in the future can the Nash point be chosen. ■

We can now summarize these results in the following theorem.

Theorem 1 *Assume that for all domains D the random process is regular. A sufficient condition for convergence to N to occur almost surely on all finite domains D for which the Nash theorem holds true is that* $0<\delta\leq\frac{\min\{a,b,c,d\}}{\min\{a,b,c,d\}+c+d}$. *This condition is also necessary if* $b\leq\min\{a,c,d\}$.

Remark 1 *The benchmark case discussed in the introduction is $a=b=c=d$, for which one obtains:*

Proposition 4 *In the benchmark case, the Judge converges to N almost surely on all finite domains D for which the Nash theorem holds true if and only if* $0<\delta\leq 1/3$.

Remark 2 *These results depend on the Judge being myopic. For instance, in Prop. 3 the Judge could anticipate that j will occur at some date and that the only way not to incur any penalty is to take $N(i)$ right from the beginning.*

4 Kalai-Smorodinsky

The Kalai-Smorodinsky solution is denoted KS. One has

$$KS(i)=\{x\in\partial i \mid x_1/x_2=I_1(i)/I_2(i)\}.$$

The Kalai-Smorodinsky theorem is said to hold on D if $KS(.)$ is the only solution satisfying WP, Sym, ScInv and IMon on D.

Let $i \in D$. Condition (KS) holds for i if there exists a sequence $j_1, \ldots j_n$ such that $j_1 = i$, j_n is symmetric and for all $t = 1, \ldots, n-1$, either

(i) $j_t \subseteq j_{t+1}$ (or $j_t \supseteq j_{t+1}$) and $I(j_t) = I(j_{t+1})$ and $KS(j_{t+1}) \in \partial j_t$; or

(ii) $\exists \alpha \in \mathbb{R}^2_{++}$, j_{t+1} is an α–rescaling of j_t.

Again, and without risk of confusion with the previous section, let us call such a sequence a *special chain beginning at* i.

Proposition 5 *The KS theorem holds on a finite domain D if and only if Condition (KS) holds for all $i \in D$.*

The proof of this proposition is tedious and is relegated to the appendix. The complication, in comparison to the proof of Prop. 1, comes from the fact that IMon can relate two problems i, j even if Condition (KS) holds for only one of them. Therefore, a solution which coincides with KS on the subdomain of problems i for which Condition (KS) holds may, via IMon, impose substantial constraints on its selections in the problems i for which Condition (KS) does not hold. It is not trivial to show that this solution may nonetheless depart from KS on some of these problems.

It is an open question whether this result is also true for an infinite domain.

Proposition 6 *Assume that the random process is regular and that Condition (KS) holds for all $i \in D$. If*

$$0 < \delta \leq \frac{\min\{a, b, c, g\}}{\min\{a, b, c, g\} + c + g}. \tag{2}$$

then the Judge converges almost surely to the Kalai-Smorodinsky solution.

Proof. The proof closely mimics the proof of Prop. 2, with IMon and g replacing Ind and d, respectively. ■

Proposition 7 *Assume that the random process is regular. If $\delta = 0$ or $\frac{b}{b+c+g} < \delta < 1$, there exists a domain D such that Condition (KS) holds for all $i \in D$ but convergence to the Kalai-Smorodinsky solution does not occur almost surely.*

Proof. The proof mimics the proof of Prop. 3, with the same example. ■

We can therefore conclude with a theorem that is very similar to what was obtained with the Nash solution.

Theorem 2 *Assume that for all domains D the random process is regular. A sufficient condition for convergence to KS to occur almost surely on all finite domains D for which the Kalai-Smorodinsky theorem holds true is that* $0 < \delta \leq \frac{\min\{a,b,c,g\}}{\min\{a,b,c,g\}+c+g}$. *This condition is also necessary if* $b \leq \min\{a, c, g\}$. *In the benchmark case $a = b = c = g$, the Judge converges to KS almost surely on all finite domains D for which the Kalai-Smorodinsky theorem holds true if and only if* $0 < \delta \leq 1/3$.

5 Egalitarian solution

The Egalitarian solution is denoted E. One has

$$E(i) = \{x \in \partial i \mid x_1 = x_2\}.$$

The egalitarian theorem is said to hold on D if $E(.)$ is the only solution satisfying WP, Sym, and Mon on D. This theorem is a variant of Th. 1 in Kalai (1977) and can be found in Thomson and Lensberg (1989, Th. 2.5) and Peters (1992, Th. 4.31).

Let $i \in D$. Condition (E) holds for i if there exists a sequence $j_1, \ldots j_n$ such that $j_1 = i$, j_n is symmetric and for all $t = 1, \ldots, n-1$, $E(j_t) = E(j_n)$ and either $j_t \subseteq j_{t+1}$ or $j_{t+1} \subseteq j_t$. Again the sequence $j_1, \ldots j_n$ will be called a special chain beginning at i.

Proposition 8 *The egalitarian theorem holds on a finite domain D if and only if Condition (E) holds for all $i \in D$.*

Proof. If: Let φ be any solution on D satisfying the axioms of the egalitarian theorem. Let $i \in D$. By Condition (E) there is a special chain $j_1, \ldots j_n$ beginning at i. By Sym,

$E(j_n)$ is chosen from j_n and one rolls back along the special chain by applying Mon. This implies $\varphi(i) = E(i)$.

Only if: Let D_+ be the subset of D containing the problems i for which Condition (E) holds. We must show that $D_+ = D$ if the egalitarian theorem holds on D. Suppose that there exists a problem $k \in D \setminus D_+$. Let

$$Z = \left\{x \in \mathbb{R}^2_+ \mid \exists i \in D_+,\ x = E(i)\right\}.$$

Construct a monotone path P from zero for which the intersection with the 45° line coincides with Z on $\mathbb{R}^2_{++}$. More precisely, P is the graph of an increasing function f such that $f(0) = 0$, and

$$\{x \in P \mid x_1 = x_2\} = Z \cup \{0\}.$$

Let φ be defined by: for all $i \in D$, $\{\varphi(i)\} = P \cap \partial i$. By construction φ satisfies WP and Mon. It satisfies Sym because all symmetric problems are in D_+, and φ coincides with E on D_+. But $\varphi \neq E$ unless $D = D_+$, which proves the "only if" part of the proposition. ■

As in the previous sections, consider a finite domain D and an infinite history $t = 1, 2, \ldots$

Proposition 9 *Assume that the random process is regular and that Condition (E) holds for all $i \in D$. If*

$$0 < \delta \leq \frac{\min\{a, b, h\}}{\min\{a, b, h\} + h},$$

then the Judge converges almost surely to the egalitarian solution.

Proof. 1. Enumerate the elements of D as $1, 2, \ldots, M$. For any element i define the special chain beginning at i as $i, j_2(i), ..., j_{n(i)}(i)$. With probability one, we will get at some point the following sequence of societies:

$$j_{n(1)}(1), j_{n(1)-1}(1), ..., 1, j_{n(2)}(2), ..., 2, j_{n(3)}(3), ..., 3, ..., j_{n(M)}(M), ..., M.$$

Suppose this sequence occurs at date T. If $E(j_{n(1)}(1))$ is not chosen, the penalty will be at least $\min\{a,b\}$, since either WP or Sym will be violated. If, however, $E(j_{n(1)}(1))$ is chosen, this will entail at most one violation with respect to all previous choices — namely, for any previous date, a violation of Mon. So the worst penalty that can be incurred is

$$h\sum_{t=1}^{T-1}\delta^t < h\sum_{t=1}^{\infty}\delta^t = h\frac{\delta}{1-\delta}.$$

The assumption

$$\delta \leq \frac{\min\{a,b,h\}}{\min\{a,b,h\}+h}$$

implies

$$\delta \leq \frac{\min\{a,b\}}{\min\{a,b\}+h} \tag{3}$$

because

$$\frac{\min\{a,b,h\}}{\min\{a,b,h\}+h} \leq \frac{\min\{a,b\}}{\min\{a,b\}+h},$$

and (3) is equivalent to

$$h\frac{\delta}{1-\delta} \leq \min\{a,b\}.$$

Therefore,

$$h\sum_{t=1}^{T-1}\delta^t < \min\{a,b\},$$

so the Judge will choose $E(j_{n(1)}(1))$.

2. Now consider the second element in the sequence. If the Judge does not choose $E(j_{n(1)-1}(1))$, he violates Mon w.r.t. the previous date, so the penalty is at least δh. If he does choose $E(j_{n(1)-1}(1))$, he is penalized at most

$$h\sum_{t=2}^{T}\delta^t < h\sum_{t=2}^{\infty}\delta^t = h\frac{\delta^2}{1-\delta}.$$

One has

$$\delta \leq \frac{\min\{a,b,h\}}{\min\{a,b,h\}+h} \leq 1/2,$$

and therefore $\delta/(1-\delta) < 1$, implying that $h\frac{\delta^2}{1-\delta} \leq \delta h$. In conclusion, as

$$h\sum_{t=2}^{T}\delta^t < \delta h,$$

the Judge chooses $E(j_{n(1)-1}(1))$.

3. In this way we see that we have the E choice on the whole sequence.

4. Now let the element that occurs after this sequence be i. If Judge does not choose $E(i)$, he violates one axiom with respect to the previous occurrence of i in the sequence — namely, Mon. The penalty is therefore at least $h\delta^t$ for some $1 \leq t \leq \sum_{j=2}^{M} n(j)+1$. (The worst case is that $i=1$.) Let $Q = \sum_{j=2}^{M} n(j)+1$. On the other hand if he chooses $E(i)$, he is penalized by less than

$$h\sum_{t=Q+1}^{\infty}\delta^t = h\frac{\delta^{Q+1}}{1-\delta}.$$

So he will choose $E(i)$ as long as

$$h\frac{\delta^{Q+1}}{1-\delta} \leq h\delta^Q,$$

which holds true as it is equivalent to $\delta \leq 1/2$.

5. By induction he chooses E henceforth. This concludes the proof. ■

Proposition 10 *Assume that the random process is regular. If $\delta = 0$ or $\frac{b}{h+b} < \delta < 1$, there exists a domain D such that Condition (E) holds for all $i \in D$ but convergence to the egalitarian solution does not occur almost surely.*

Proof. Let $D = \{i, j\}$, as described in Fig. 1. The problem j is symmetric.

Suppose $\frac{b}{h+b} < \delta < 1$. The first inequality is equivalent to

$$\frac{b}{h}\left(\frac{1-\delta}{\delta}\right) < 1.$$

There is a positive probability that history starts with T occurrences of i for T satisfying

$$T > \frac{\ln\left(1-\frac{b}{h}\left(\frac{1-\delta}{\delta}\right)\right)}{\ln\delta}.$$

Suppose the Judge picks point x for $t = 1, ..., T$.

Let j occur at $t = T+1$. If the Judge chooses $E(j)$ the penalty is $h\sum_{t=1}^{T}\delta^t$. If he chooses x the penalty is b. If he chooses another point the penalty is either a or b, plus $h\sum_{t=1}^{T}\delta^t$. This last option is therefore dominated by $E(j)$. Noting that

$$h\sum_{t=1}^{T}\delta^t = h\delta\frac{1-\delta^T}{1-\delta},$$

we have

$$\begin{aligned}
h\sum_{t=1}^{T}\delta^t &> b \Leftrightarrow 1-\delta^T > \frac{b}{h}\frac{1-\delta}{\delta} \\
&\Leftrightarrow \ln\left(1-\frac{b}{h}\left(\frac{1-\delta}{\delta}\right)\right) > T\ln\delta \\
&\Leftrightarrow T > \frac{\ln\left(1-\frac{b}{h}\left(\frac{1-\delta}{\delta}\right)\right)}{\ln\delta},
\end{aligned}$$

which is true by assumption. Therefore the judge chooses x.

Consider a period $S > T+1$ and assume that x has been chosen at all times before (we know this to be true for $S = T+2$). If i occurs, x is picked again without any penalty while any other point costs a penalty. If j occurs, choosing x costs b while choosing $E(j)$ costs $h\sum_{t=1}^{T+1}\delta^t > h\sum_{t=1}^{T}\delta^t$. So, again x is chosen.

By induction, at no period in the future can the egalitarian point be chosen. ■

Theorem 3 *Assume that for all domains D the random process is regular. A sufficient condition for convergence to E to occur almost surely on all finite domains D for which the egalitarian theorem holds true is that* $0 < \delta \leq \frac{\min\{a,b,h\}}{\min\{a,b,h\}+h}$. *This condition is also necessary if* $b \leq \min\{a, h\}$. *In the benchmark case* $a = b = h$, *the Judge converges to E almost surely on all finite domains D for which the Kalai-Smorodinsky theorem holds true if and only if* $0 < \delta \leq 1/2$.

6 Conclusion

An interesting fact is that, in the benchmark case, we get convergence to the solution in question (Nash, Kalai-Smorodinsky, Egalitarian) precisely when discounting the future is large. This is somewhat counterintuitive: one might have thought that convergence to the solution would occur only if the past was quite important, but it is in fact just the opposite.

Suppose we have a set $\mathfrak{T}$ of axiomatic theorems of the type we have discussed here, and for each theorem $\tau \in \mathfrak{T}$ we prove that in the benchmark case, almost-sure convergence to the appropriate solution occurs if and only if $\delta \in (0, \delta_\tau]$. This provides a way ranking the axiomatic theorems in terms of plausibility: the greater is δ_τ, the more plausible is the theorem, in the sense that the dynamic version of the theorem (as developed here) holds for a larger set of discount factors. Thus, we would say that the egalitarian theorem is more plausible than Nash's theorem or Kalai and Smorodinsky's.

To be precise, we are saying that if we observe societies which abide by an egalitarian constitution and societies which abide by a Nash constitution, and discount factors vary across societies randomly, then it is more likely that we will observe allocations looking like the egalitarian solution in the egalitarian societies than allocations looking like the Nash solution in Nash societies, because $(0, 1/3] \subset (0, 1/2]$.

We conclude with an anecdote. Many years ago, one of us (Roemer) interviewed a number of functionaries in the World Health Organization (WHO) about how they allocated their budget among member countries, a process that WHO undertakes biennially. It is not too difficult to think of what country preferences (or utility functions) mean in this context, something that is quite clearly specified in WHO documents. (It is a measure of average health status of persons in the country.) The resource is given by the WHO budget. It is possible to characterize the lexicographic minimum solution ('leximin') by a small number of axioms, and when interviewed, the functionaries all agreed that they would certainly respect these axioms in the budget allocation process. Documents issued by the Organization also stipulated 'norms' for budget allocation which look very much like some of these axioms. Yet an examination of the history of budget allocations shows that nothing like the leximin solution is followed. One possible explanation is that a given administration lasts only for a small number of years, and so judicial precedent does not have time to bite. It is conceivable that the discounting of penalties is not as we have postulated: for the period of the adminisration in power, δ may be quite large, but it becomes zero for dates occurring in earlier administrations. In this case, even if the correct rule is eventually adopted by an administration, the next administration is not constrained by it and may start a new history. A second possible explanation is that, although the functionaries would have liked to implement leximin, they were overruled by the Assembly of WHO, a political body where country representatives fight for all they can get. Not having the present theory to guide him at that time, it did not occur to the co-author to study this question.

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Appendix

Proof of Prop. 5. If: Let φ be any solution on D satisfying the axioms WP, Sym, ScInv and IMon. Let $i \in D$. By Condition (KS) there is a special chain $j_1, \ldots j_n$ beginning at i. By Sym, $\varphi(j_n) = KS(j_n)$ and one can roll back along the special chain to i, and at each step $\varphi(j_k) = KS(j_k)$ either by IMon (case (i)) or by ScInv (case (ii)). For $k = 1$, we have $\varphi(i) = KS(i)$. It follows that $\varphi = KS$ on D.

Only if: 1. For all solutions φ and all problems $k, i \in D$ such that either $k \subseteq i$ or $k \supseteq i$, let $C_\varphi(k;i)$ denote the constraint imposed on $\varphi(k)$ by $\varphi(i)$, IMon and WP, i.e., $C_\varphi(k;i)$ is the subset of ∂k such that IMon is not violated if $\varphi(k) \in C_\varphi(k;i)$, given $\varphi(i)$. Specifically, there are eight cases:

(i) $I(k) = I(i)$ and $\varphi(i) \in \partial k$. Then $C_\varphi(k;i) = \{\varphi(i)\}$.

(ii) $I(k) = I(i)$ and $\varphi(i) \in k \setminus \partial k$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x \geq \varphi(i)\}$.

(iii) $I(k) = I(i)$ and $\varphi(i) \notin k$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x \leq \varphi(i)\}$.

(iv) $I_1(k) = I_1(i)$ and $I_2(k) < I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_2 \leq \varphi_2(i)\}$.

(v) $I_1(k) = I_1(i)$ and $I_2(k) > I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_2 \geq \varphi_2(i)\}$.

(vi) $I_1(k) < I_1(i)$ and $I_2(k) = I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_1 \leq \varphi_1(i)\}$.

(vii) $I_1(k) > I_1(i)$ and $I_2(k) = I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_1 \geq \varphi_1(i)\}$.

(viii) $I_1(k) \neq I_1(i)$ and $I_2(k) \neq I_2(i)$. Then $C_\varphi(k;i) = \partial k$.

Note that if φ satisfies WP and $\varphi(k) \in C_\varphi(k;i)$, necessarily $\varphi(i) \in C_\varphi(i;k)$. This can be checked for each case:

(i) This also corresponds to case (i) for i : $\varphi(k) = \varphi(i)$ and $C_\varphi(i;k) = C_\varphi(k;i)$.

(ii) This corresponds to case (iii) for i : As $\varphi(i) \in k \setminus \partial k$, $\varphi(k) > \varphi(i)$, $\varphi(k) \notin i$ and $C_\varphi(i;k) = \{x \in \partial i \mid x \leq \varphi(k)\}$.

(iii) This corresponds to case (ii) for i : As $\varphi(i) \notin k$, $\varphi(k) < \varphi(i)$, $\varphi(k) \in i \setminus \partial i$ and $C_\varphi(i;k) = \{x \in \partial i \mid x \geq \varphi(k)\}$.

(iv) This corresponds to case (v) for i : $\varphi_2(k) \leq \varphi_2(i)$ and $C_\varphi(i;k) = \{x \in \partial i \mid x_2 \geq \varphi_2(k)\}$.

(v) This corresponds to case (iv) for i : $\varphi_2(k) \geq \varphi_2(i)$ and $C_\varphi(i;k) = \{x \in \partial i \mid x_2 \leq \varphi_2(k)\}$.

(vi),(vii) are treated similarly.

(viii): This also corresponds to case (viii) for i and $C_\varphi(i;k) = \partial i$.

2. Let D_+ denote the subset of D containing the problems i for which Condition (KS) holds. We must show that $D_+ = D$ if the Kalai-Smorodinsky theorem holds on D. Suppose that $D \setminus D_+$ is not empty. For every $k \in D \setminus D_+$:

1) k is not symmetric because symmetric problems are in D_+;

2) k is not a rescaling of a problem $i \in D_+$ because otherwise $k, i, ...$ starts a special chain beginning at k, implying that $k \in D_+$;

3) k can be related by IMon to a problem $i \in D_+$. But necessarily, either $I(k) \neq I(i)$ or $KS(i) \notin \partial k$; otherwise, a special chain beginning at k can be formed and thus $k \in D_+$.

The set $D \setminus D_+$ can be partitioned into the equivalence classes of the equivalence relation "is a rescaling of". Say that two equivalence classes E, E' are "linked" if there is $i \in E$, $i' \in E'$ such that $I(i) = I(i')$, $KS(i) = KS(i')$, and either $i \subseteq i'$ or $i \supseteq i'$.

The relation "is linked to" is not transitive in general because it may happen that, for a particular triple E, E', E'', E and E' are linked via two problems i, i', E' and E'' are linked via j', j'', and E and E'' are not linked. We will also be interested in its transitive closure, called "directly or indirectly linked to".

3. Pick one particular equivalence class E^* (for the rescaling relation) and all the equivalence classes that are directly or indirectly linked to it. Call the union of these classes D^*. This is a subset of $D \setminus D_+$ (not necessarily a proper subset).

Pick a member of E^*, i^*. E^* and i^* will play a special role in the rest of the proof. If $D^* \neq E^*$, let E be any other equivalence class in D^*. Consider first the case in which the equivalence classes E and E^* are linked by two problems $i \in E$, $j^* \in E^*$. Therefore there exists (not necessarily in E) a rescaling of i, denoted i_E, such that $I(i^*) = I(i_E)$, $KS(i^*) = KS(i_E)$, and either $i^* \subseteq i_E$ or $i^* \supseteq i_E$. Consider now the case in which E and E^* are only indirectly linked (which means that they are not linked but are directly or indirectly linked). One can then pick an arbitrary i in E and construct a rescaling of i, denoted i_E, such that $I(i^*) = I(i_E)$. (In this case there is no guarantee that $KS(i^*) = KS(i_E)$, $i^* \subseteq i_E$ or $i^* \supseteq i_E$.)

For every equivalence class E in D^*, one can construct such a i_E following the approach described for each of the two cases in the previous paragraph. The problems i_E may or may not belong to D^*. Let D^{**} be the (possibly empty) subset of these problems i_E that do not belong to D^*. Note that $D^{**} \cap D_+ = \varnothing$, because if one had $i_E \in D_+$, any $k \in E$, being a rescaling of i_E, would then be the beginning of a special chain, contradicting the fact that $E \cap D_+ = \varnothing$.

4. Consider any two problems $k \in D^*$, $i \in D \setminus D^*$. Suppose that k is related to i by IMon, which implies that $k \subseteq i$ or $k \supseteq i$. Necessarily, either $I(k) \neq I(i)$ or $KS(i) \notin \partial k$, as we now show. First, suppose that $i \in D_+$. Then $I(k) = I(i)$ and $KS(i) \in \partial k$ would imply that $k \in D_+$, a contradiction. Second, suppose that $i \in D \setminus D_+$. In this case, $I(k) = I(i)$ and $KS(i) \in \partial k$ would imply that the equivalence classes of k and i are

linked, contradicting the fact that $k \in D^*$ and $i \in D \setminus D^*$.

We now derive consequences from the fact that either $I(k) \neq I(i)$ or $KS(i) \notin \partial k$ whenever $k \in D^*$ and $i \in D \setminus D^*$ are related by IMon. Consider first the case $I(k) \neq I(i)$. Four subcases are possible: $C_{KS}(k;i) = \{x \in \partial k \mid x_2 \leq KS_2(i)\}$, $C_{KS}(k;i) = \{x \in \partial k \mid x_2 \geq KS_2(i)\}$, $C_{KS}(k;i) = \{x \in \partial k \mid x_1 \leq KS_1(i)\}$, $C_{KS}(k;i) = \{x \in \partial k \mid x_1 \geq KS_1(i)\}$. Focus on the first subcase, the other subcases being similar. Because this first subcase corresponds to $k \subseteq i$, $I_1(k) = I_1(i)$ and $I_2(k) < I_2(i)$, one then has $KS_2(k)/KS_1(k) < KS_2(i)/KS_1(i)$. As $k \subseteq i$, the point $\hat{x} \in \partial k$ such that $\hat{x}_2 = KS_2(i)$ (which belongs to $C_{KS}(k;i)$) satisfies $\hat{x}_1 \leq KS_1(i)$ and thus $\hat{x}_2/\hat{x}_1 \geq KS_2(i)/KS_1(i)$. Therefore $KS(k)$, which is obviously an element of $C_{KS}(k;i)$, satisfies $KS_2(k)/KS_1(k) < \hat{x}_2/\hat{x}_1$ and is not an extreme point of $C_{KS}(k;i)$. Consider the second case, $KS(i) \notin \partial k$. As the subcase in which $I(k) \neq I(i)$ has already been examined, we can focus on the subcase in which $I(k) = I(i)$. One then has either $KS(k) \gg KS(i)$ or $KS(k) \ll KS(i)$ and again $KS(k)$ is not an extreme point of $C_{KS}(k;i)$.

5. For every $k \in D^*$, let $C(k) = \bigcap_{i \in D \setminus D^*} C_{KS}(k;i)$. This set contains $KS(k)$ and $KS(k)$ is not an extreme point of it, because these two properties are satisfied by each of the $C_{KS}(k;i)$, of which there is a finite number.

Take any $k^* \in D^*$. Let E be its equivalence class. For every $k \in E$, there is $\alpha \in \mathbb{R}^2_{++}$ such that k^* is an α-rescaling of k. Let $C|_{k^*}(k)$ denote the α-rescaling of $C(k)$. This is a subset of ∂k^*. Note that $KS(k^*)$ is an α-rescaling of $KS(k)$. As $KS(k)$ is a non-extreme element of $C(k)$, then $KS(k^*)$ is a non-extreme element of $C|_{k^*}(k)$. Therefore the subset $\bigcap_{k \in E} C|_{k^*}(k)$ contains $KS(k^*)$ and $KS(k^*)$ is not an extreme point of this subset. Let $C^*(k^*)$ denote this subset. Note that when k is a rescaling of k', $C^*(k)$ is a rescaling of $C^*(k')$.

Take any $i_E \in D^{**}$, where E is an equivalence class in D^*. Recall that $i_E \notin D^*$ but i_E is a rescaling of each member of E. The subset $\bigcap_{k \in E} C|_{i_E}(k)$ contains $KS(i_E)$ and $KS(i_E)$ is not an extreme point of this subset. Let $C^*(i_E)$ denote this subset. Note that for all $k \in E$, $C^*(i_E)$ is a rescaling of $C^*(k)$.

6. Now let us look again at the particular i^* and all the i_E (that may belong to D^* or D^{**}) that were introduced in step 3. One has $I(i^*) = I(i_E)$ and therefore $KS_2(i^*)/KS_1(i^*) = KS_2(i_E)/KS_1(i_E)$ for all E in D^*, while $KS(i^*)$ is not an extreme point of $C^*(i^*)$ just as $KS(i_E)$ is not an extreme point of $C^*(i_E)$. Therefore there is $\mu \in \mathbb{R}^2_{++}$, $\mu \neq KS_2(i^*)/KS_1(i^*)$ such that the point $x \in \partial i^*$ such that $x_2/x_1 = \mu$ belongs to $C^*(i^*)$ and for all E in D^*, the point $x \in \partial i_E$ such that $x_2/x_1 = \mu$ belongs to $C^*(i_E)$.

Now we are ready to define a solution φ as follows. On $D \setminus D^*$, it coincides with KS. For $k \in D^*$, there is $\alpha \in \mathbb{R}^2_{++}$ and E in D^* such that k is an α-rescaling of i_E (or i^*); then $\varphi(k)$ is the point $x \in \partial k$ such that $x_2/x_1 = (\alpha_2/\alpha_1)\mu$. Note that, as $C^*(k)$ is a α-rescaling of $C^*(i_E)$ (or of $C^*(i^*)$), this implies that $\varphi(k) \in C^*(k)$.

7. It is obvious that φ satisfies WP, Sym and ScInv. It also obviously satisfies IMon on $D \setminus D^*$.

Consider two problems $k \in D^*$, $i \in D \setminus D^*$ that are related by IMon. By construction, $\varphi(k) \in C^*(k) \subset C(k) \subset C_{KS}(k;i) = C_\varphi(k;i)$, where the last equality is due to $\varphi(i) = KS(i)$. And conversely this implies $\varphi(i) \in C_\varphi(i;k)$.

Consider two problems $i, k \in D^*$ that are related by IMon. First case: If they belong to the same equivalence class, IMon is satisfied because it is implied by ScInv in this case. Second case: Suppose i is a rescaling of i_E, k a rescaling of $i_{E'}$ (one problem among $i_E, i_{E'}$ may be i^*). One has $\varphi_2(i_E)/\varphi_1(i_E) = \varphi_2(i_{E'})/\varphi_1(i_{E'}) = \mu$. Without loss of generality, suppose that $i \subseteq k$ and $I_1(i) = I_1(k)$. If $I_2(i) = I_2(k)$, then i and k are α-rescalings of i_E and $i_{E'}$, respectively, for the same α (recall that $I(i_E) = I(i_{E'}) = I(i^*)$). Then $\varphi_2(i)/\varphi_1(i) = \varphi_2(k)/\varphi_1(k) = (\alpha_2/\alpha_1)\mu$ and IMon is satisfied. If $I_2(i) < I_2(k)$, then $\varphi_2(i)/\varphi_1(i) = (\alpha_2/\alpha_1)\mu$ and $\varphi_2(k)/\varphi_1(k) = (\alpha_2'/\alpha_1)\mu$ for some $\alpha_1, \alpha_2, \alpha_2'$ such that $\alpha_2 < \alpha_2'$. IMon would be violated if one had $\varphi_2(k) < \varphi_2(i)$. This inequality is equivalent to

$$\varphi_2(k) = \varphi_1(k)(\alpha_2'/\alpha_1)\mu < \varphi_1(i)(\alpha_2/\alpha_1)\mu = \varphi_2(i),$$

$$\varphi_1(k)(\alpha_2'/\alpha_2) < \varphi_1(i).$$

One would then obtain $\varphi(i) \gg \varphi(k)$, contradicting the fact that $i \subseteq k$ and that φ satisfies WP. Therefore IMon is satisfied.

The solution φ coincides with KS only on $D \setminus D^*$, which shows that if $D \setminus D_+$ is not empty, the Kalai-Smorodinsky theorem does not hold on D. This achieves the proof of the "only if" part of the proposition. ■