Overcoming Ideological Bias in Elections*

Vijay Krishna — Penn State University

John Morgan — UC Berkeley

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Abstract

We study a model in which voters choose between two candidates on the basis of both ideology and competence. While the ideology of the candidates is commonly known, voters are imperfectly informed about competence. Voter preferences, however, are such that it is a dominant strategy to vote according to ideology alone. When voting is compulsory, the candidate of the majority ideology prevails and this may be inefficient from a social perspective. However, when voting is voluntary and costly, we show that turnout adjusts endogenously so that the outcome of a large election is always first-best.

1 Introduction

In what may justly be called the "first welfare theorem" of political economy, Condorcet's (1785) celebrated Jury Theorem says that if voters have common interests but dispersed information, sincere voting under majority rule results in efficient outcomes. Like its analog in economics, Condorcet's result argues in favor of decentralized decision making. The Jury Theorem has been considerably generalized since Condorcet's original formulation. It holds when voters are strategic or when election rules require a supermajority (Feddersen and Pesendorfer, 1998). The result has also influenced the thinking of legal scholars such as Sunstein (2009).

Condorcet's formulation presumes that information is the primary hurdle to effective group decision making—information is dispersed and majority voting is an effective means of collecting this information and translating it into a decision. This, however, ignores the role played by ideology. When ideology is important, it is easy to see that the Condorcet Jury Theorem no longer works as advertised. Consider a two-candidate election where voters care both about the candidates' competence (a common component) and their ideology (a private component). While the ideology of each candidate is commonly known, information about competence is dispersed. If, for each voter, ideology outweighs competence—an incompetent candidate of the correct ideology is favored over a competent candidate of the opposing ideology—then

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voting is purely along ideological lines. As a result, the candidate of the majority ideology will win even if incompetent, and this may be inefficient. Society may be better served by choosing the more competent candidate of the minority ideology. When ideology is important, information does not aggregate, and the "first welfare theorem" fails.[1]

The argument above rests on the implicit assumption that all eligible voters actually vote or, more generally, that participation is exogenous. In this paper, we argue that the failure of the first welfare theorem stems entirely from this rather unrealistic assumption. When participation is endogenous—voting is voluntary and costly—then large majoritarian elections are always efficient (Theorem 1). As an example, suppose that voters of one ideology, say A, hold a two-to-one advantage in numbers over voters of the other ideology, say B. Even though voting is purely ideology-based, the theorem says that in large elections B voters will turn out at no less than *twice* the rate of A voters if and only if it is socially efficient for B to win. Thus, endogenous participation neutralizes the corrosive influence of ideology and the first welfare theorem is restored.

Our main result generalizes the Condorcet Jury Theorem to a setting in which ideological considerations are paramount. Our model is, in a sense, a worst case scenario for the theorem—ideology is so important that it dominates all other considerations for individuals once in the polling place. But, as we show, in equilibrium, turnout always adjusts so as to restore efficiency. Since voting in our model is costly, majoritarian elections with endogenous participation also have the desirable property of inducing efficient sorting. In short, we show that large majoritarian elections produce first-best outcomes. Before placing the result in the context of the broader literature, it is useful to illustrate it by means of a simple example.

An Example[2] How voluntary voting can overcome the problems of ideology may be seen in the following simple setting. Suppose that there are two candidates A and B who differ in ideology and competence. With probability $\lambda > \frac{1}{2}$, a voter favors A on ideological grounds and with probability $1 - \lambda$, favors B. However, all voters know that B is the more competent candidate. Let $v > 0$ denote the gain to a type A voter (one who favors A on ideological grounds) from electing A versus B and let $V \leq 1$ denote the gain to a type B voter from electing B versus A. Since B is the more competent candidate, it is natural to suppose that $V > v$.

Clearly, when voters go to the polls, they will vote solely on ideological grounds. Type A voters will vote for A and type B voters for B. If voting is compulsory, so that all eligible voters vote, the ratio of the expected number of votes for A versus B will be $\lambda/(1-\lambda) > 1$. In large elections, A will win with probability close to one. If it is the case that $\lambda v < (1-\lambda)V$, however, the election outcome would not be efficient since social welfare would be higher if B were to be chosen.

Now suppose that voting is voluntary and costly. Each voter has a privately known cost of voting $c \in [0, 1]$ which is drawn independently from a uniform distribution.

[1] Proposition 3.1 below formalizes this idea.
[2] Andy Postlewaite suggested looking at this special case.

A voter participates if and only if the benefits of voting exceed his voting cost. Let p_A and p_B denote the participation probabilities (turnout rates) of the two types of voters. Equilibrium dictates that these satisfy

$$\begin{aligned} p_A &= v \Pr[Piv_A] \\ p_B &= V \Pr[Piv_B] \end{aligned}$$

where $\Pr[Piv_A]$ is the probability that an additional vote for A is pivotal (affects the outcome). Similarly, $\Pr[Piv_B]$ is the probability that a vote for B is pivotal. The first equilibrium condition says that the expected benefit to a type A voter must equal the cost threshold for participation. Since costs are uniformly distributed, this cost threshold is the same as the participation probability. The equilibrium condition for type B is analogous.

The equilibrium conditions imply the ratio of vote shares is proportional to the ratio of the relative welfare gains and the proportionality factor is the ratio of pivotal probabilities, that is,

$$\underbrace{\frac{\lambda p_A}{(1-\lambda) p_B}}_{\text{Vote ratio}} = \underbrace{\frac{\lambda v}{(1-\lambda) V}}_{\text{Welfare ratio}} \times \underbrace{\frac{\Pr[Piv_A]}{\Pr[Piv_B]}}_{\text{Pivot ratio}}$$

First, suppose that it is efficient to elect B; that is, the welfare ratio is less than one. We claim that in that case, the vote ratio must also be less than one, that is, it should favor B. If the vote ratio favored A, then B is more likely to be behind by exactly one vote than ahead by one vote. Thus whenever the vote ratio is *greater* than one, the pivot ratio is *less* than one.[3] But since the welfare ratio is also less than one, this leads to a contradiction. Hence, when it is efficient to elect B, the vote ratio must favor B.

Next, suppose that it efficient to elect A; that is, the welfare ratio is greater than one. We claim that the vote ratio must also be greater than one, that is, the vote ratio should favor A. If the vote ratio favored B, the pivot ratio would be *greater* than one. But since the welfare ratio is also greater than one, this again leads to a contradiction. When it is efficient to elect A, the vote ratio must favor A.

One might surmise that voluntary voting succeeds because the private benefits of voting align perfectly with social welfare. Were that the case, one would expect that the vote ratio and the welfare ratio would be equal. This, however, is not true in equilibrium. While the vote ratio "mirrors" the welfare ratio in the sense that both lie on the same side of 1, the election is closer than the welfare considerations would imply; that is, the vote ratio is closer to 1 than is the welfare ratio.

While the example illustrates how endogenous turnout can, in principle, restore efficiency, its simplicity derives from ignoring the key problem that the election is supposed to solve—information aggregation. In general, competence is not commonly

[3] The pivot probability $\Pr[Piv_A]$ is the sum of the probability that there is a tie and the probability that A is one vote short. $\Pr[Piv_B]$ is similarly defined. Since $\lambda p_A \geq (1-\lambda) p_B$, the probability that A is one vote short is smaller than the probability that B is one vote short.

known, thus, the turnout calculus is influenced by the voter's perceived probability of a candidate's competence. When there is uncertainty about which is the competent candidate, the vote ratio need not mirror the welfare ratio—a majority of those voting can favor an incompetent but ideologically preferred candidate even when this is not socially optimal (see Example 5.1 below). The situation is corrected, however, as the size of the electorate increases. In large elections, the vote ratio mirrors the welfare ratio and the correct candidate is elected with probability one. One may surmise that this is because in large elections, the competence of the candidates becomes known with high probability. This is not the case as residual uncertainly about competence remains even in the limit. Thus, the process of generalizing the example requires considerable care.

Literature Modern analyses of the Jury Theorem stress the fact that when voters have *common* interests, sincere voting is inconsistent with equilibrium (Austen-Smith and Banks, 1996). Feddersen and Pesendorfer (1998) show that Condorcet's result still holds if voters are strategic—in large elections, ("insincere") equilibria of the resulting voting game aggregate information perfectly. They also show that the Jury Theorem extends to all supermajority rules (except unanimity). McLennan (1998) shows that these results are a consequence of the fact that the Condorcet model is a game of common interest and so always has Pareto efficient equilibria. In all of these models it is implicitly assumed that everyone votes, that is, voting is compulsory. In an earlier paper (Krishna and Morgan, 2009), we show that, with common interests, costly and voluntary voting results in sincere voting as an equilibrium and that such equilibria are welfare superior to those with compulsory voting.

Palfrey and Rosenthal (1985) are the first to study costly voting but in a model with pure *private* values—that is, voters care only about the ideology of the candidates. Börgers (2004) compares compulsory and voluntary voting in this setting. He shows that voluntary voting, by economizing on voting costs, is superior to compulsory voting. Krasa and Polborn (2009) show that Börgers' result may not hold when the electorate is ideologically biased, that is, there is a majority that favors one of the candidates. Taylor and Yildirim (2008) investigate the asymptotic properties of participation in a model with private values when voting is costly.

Feddersen and Pesendorfer (1997) study a model in which voter preferences have *both* private and common components and these are dispersed in the population. Under compulsory voting, if there are enough "centrists"—voters who care more about the common component than the private—then efficiency obtains. While our model shares many of these features, it assumes that ideology is dominant in voters' minds—there are no centrists. As a result, if voting is compulsory, efficiency surely fails.[4] But we show that voluntary voting resolves the problem.

In related work, Ghosal and Lockwood (2009) analyze a symmetric situation with both private and common components. They compare voluntary and compulsory voting and show that, unlike in Börgers' model, voluntary voting may produce too

[4]Our model thus fails Bhattacharya's (2008) necessary and sufficient condition for information aggregation under compulsory voting.

little or too much turnout. Our model differs from theirs in two respects. First, in their symmetric environment, there is no ideological bias since there are equal numbers of voters who favor each candidate on ideological grounds. In our model, overcoming the ideological bias is the key result. Second, Ghosal and Lockwood (2009) focus on turnout in "small" elections while we are concerned with efficiency in large elections.

Groseclose (2001), as well as Hummel (2010), study situations in which candidates can choose to position themselves on the ideological spectrum. Voters care both about ideology and competence (or valence). Their main concern is not about voter behavior but rather the choice of ideology by the candidates.

All of this work postulates a fixed and commonly known number of voters. Myerson (1998 & 2000) argues that precise knowledge of the number of eligible voters is an idealization at best, and suggests an alternative model in which the size of the electorate is a Poisson random variable. He then studies equilibrium behavior as the number of expected voters increases, and exhibits information aggregation results parallel to those derived in the known population models. In this paper, we also use Myerson's Poisson framework. Other work in the Poisson setting includes Feddersen and Pesendorfer (1999), who use the Poisson model to study abstention when voting is costless but preferences are diverse. In large elections, the fraction of informative (as opposed to ideological) voters goes to zero; however, information still aggregates. Herrera and Morelli (2009) also use a diverse preference Poisson model to compare turnout rates in proportional and winner-take-all parliamentary elections.

2 Preliminaries

Two candidates, A and B, compete in a majoritarian election with ties to be decided by a coin toss. Candidates differ both in their *ideology* and their *competence*. While voters are perfectly informed about each candidate's ideology, competence is not common knowledge.

Voters are of two types: One type, labeled A, favors candidate A on ideological grounds while the other, labeled B, favors candidate B. A voter's type is A with probability λ such that $\frac{1}{2} < \lambda < 1$, independent of the state. Thus there is an asymmetry between the types—a voter is more likely to favor A on ideological grounds.

Payoffs Voters' payoffs are also affected by the competence of the elected candidate. This is determined by the realized state, α or β, unknown to the voters. In state α, which occurs with probability π, candidate A is the more competent candidate while in state β, which occurs with probability $1 - \pi$, candidate B is more competent. Regardless of ideology, a voter benefits from electing the more competent candidate. The combination of a voter's type, the elected candidate, and the realized state then determine a voter's payoffs. The payoffs of type A voters, u_A, from the different outcomes are assumed to satisfy

$$u_A(A, \alpha) > u_A(A, \beta) \geq u_A(B, \beta) > u_A(B, \alpha) \tag{1}$$

The first inequality, that the payoff from electing A in state α, $u_A(A,\alpha)$, is greater than the payoff from electing A in state β, $u_A(A,\beta)$, stems purely from competence considerations—the ideologically favored candidate A is more competent in state α than in state β. The third inequality, $u_A(B,\beta) > u_A(B,\alpha)$, also stems from competence considerations alone—if the ideologically opposed candidate B is elected, it is better that this happen in state β, in which he is competent, than in state α, in which he is not. The comparison of $u_A(A,\beta)$ and $u_A(B,\beta)$, however, represents a trade-off between ideology and competence. Here, we assume that ideology trumps competence (at least weakly so).

Similar considerations apply to the payoffs, u_B, of type B voters and so we have

$$u_B(B,\beta) > u_B(B,\alpha) \geq u_B(A,\alpha) > u_B(A,\beta)$$

We assume that the payoffs of the two types are symmetric so that: $u_A(A,\alpha) = u_B(B,\beta)$; $u_A(A,\beta) = u_B(B,\alpha)$; $u_A(B,\beta) = u_B(A,\alpha)$ and $u_A(B,\alpha) = u_B(A,\beta)$.

Let $V = u_A(A,\alpha) - u_A(B,\alpha)$ denote the difference in payoffs from having the ideologically favored candidate elected in the state in which he is competent and the disfavored candidate elected when he is incompetent. Let $v = u_A(A,\beta) - u_A(B,\beta)$ be the payoff difference between having an ideologically favored incompetent candidate elected and a disfavored but competent candidate elected. Thus, v represents the trade-off between ideology and competence while V involves no such trade-off. Thus $V > v \geq 0$. Note that by symmetry, the differences are the same for type B voters.

Information Every voter receives a noisy signal, a or b, about the state. Signals are informative but noisy. Specifically, $\frac{1}{2} < \Pr[a \mid \alpha] < 1$ and $\frac{1}{2} < \Pr[b \mid \beta] < 1$. Let q_a be the *posterior* probability of state α conditional on having received an a signal:

$$q_a \equiv \Pr[\alpha \mid a] = \frac{\pi \Pr[a \mid \alpha]}{\pi \Pr[a \mid \alpha] + (1-\pi)\Pr[a \mid \beta]} \tag{2}$$

Let q_b the posterior probability of β conditional on having received a b signal:

$$q_b \equiv \Pr[\beta \mid b] = \frac{(1-\pi)\Pr[b \mid \beta]}{\pi \Pr[b \mid \alpha] + (1-\pi)\Pr[b \mid \beta]} \tag{3}$$

Note that

$$q_a + q_b > 1 \tag{4}$$

Define

$$\begin{aligned} r &= q_a \Pr[a \mid \alpha] + (1-q_b)\Pr[b \mid \alpha] \\ s &= (1-q_a)\Pr[a \mid \beta] + q_b \Pr[b \mid \beta] \end{aligned} \tag{5}$$

to be the *pre-posteriors* in states α and β, respectively. Thus, r is the probability that a voter assigns to α given that the state is α, before receiving any signals. Similarly, s is the probability that a voter assigns to β given that the state β. We will assume that the signal structure is accurate enough so that both r and s are *greater* than

$\frac{1}{2}$. In other words, on average a voter views α as the more likely state, when α is the true state. Likewise for β. Notice that if $\Pr[a \mid \alpha]$ and $\Pr[b \mid \beta]$ are high enough, then this assumption is automatically satisfied. It is also satisfied if π is close to $\frac{1}{2}$.

Voters can thus be grouped into two classes: *Unconflicted* voters are those whose signals correspond with their ideologies; these may be labelled Aa and Bb. *Conflicted* voters, labelled Ab and Ba, have signals that do not correspond with their ideologies.

Following Myerson (1998, 2000), we assume that the number of voters is a Poisson random variable N with expectation n. Thus, the probability that there are exactly k eligible voters $\Pr[N = k] = e^{-n} n^{-k}/k!$. Let σ_A be the *expected* number of votes for A in state α, and let σ_B be the expected number of votes for B in state α. Analogously, let τ_A and τ_B be the expected number of votes for A and B, respectively, in state β. Since abstention may be possible, it is only required that $\sigma_A + \sigma_B \leq n$ and $\tau_A + \tau_B \leq n$.

Voting behavior in our model is very simple. Since $u_A(A, \alpha) > u_A(B, \alpha)$ and $u_A(A, \beta) \geq u_A(B, \beta)$, it is a dominant strategy for type A voters to vote for A. Similarly, it is a dominant strategy for type B voters to vote for B.

Proposition 2.1 *It is a dominant strategy for voters to vote according to their ideologies alone; that is, type A voters vote for A and type B voters vote for B, regardless of the signals they have received.*

Note that even if ideology and competence are equally important to voters, that is, $v = 0$, competence plays no role in choosing between the candidates. Voting is driven solely by ideology.

3 Compulsory Voting

In this section, we study the welfare implications of compulsory voting. Under compulsory voting, all voters show up at the polls and given Proposition 2.1, they have a strict incentive to vote for their ideologically favored candidate. Characterizing the outcome of large elections is straightforward. In the limit as n increases, candidate A receives a share $\lambda > \frac{1}{2}$ of the votes and hence always wins, no matter what the state. Is this socially optimal?

In state α, it is, of course, always optimal to elect A. To see this, note that since $\lambda > \frac{1}{2}$ and $V > v$,

$$\lambda V > (1 - \lambda) v$$

The left-hand side of the inequality is the benefit to type A voters from electing the competent candidate A versus the incompetent candidate B whereas the right-hand side is the loss to type B voters from electing A versus B. In state β, it is optimal to elect B only if

$$\lambda v < (1 - \lambda) V \tag{6}$$

The left-hand side of the inequality is the loss to type A voters from electing B, the competent candidate with the opposing ideology, whereas the right-hand side is the benefit to type B voters from electing the competent candidate B versus the incompetent candidate A.

We will say that *competence is efficient* if (6) holds. Otherwise, we say that *ideology is efficient.*

Since compulsory voting results in A being elected in both states (when n is large), it then follows that:

Proposition 3.1 *Suppose competence is efficient. Then in large elections, compulsory voting is not welfare optimal.*

4 Voluntary Voting

We now suppose that voting is voluntary—showing up at the polls is optional—and costly. Suppose that each voter has a privately known cost of voting c_i which is independently drawn from a strictly increasing distribution function F with support $[0, \omega]$ where $\omega \geq V$. This support guarantees positive, but not full, participation. We make the assumption that F has a strictly *positive* but finite density at zero, that is, $0 < F'(0) < \infty$.

Note that even under voluntary voting, Proposition 2.1 still applies—it is a dominant strategy for each voter to vote according to ideology alone (assuming he or she turns up to vote). Since voting behavior is the same as in the case of compulsory voting, one may surmise that the conclusion of Proposition 3.1 is unaltered. After all, voters do not bring any information about the state to bear upon their voting decisions. But with voluntary voting, each voter has another decision to make—whether to vote at all—and this is the result of weighing the benefits of voting against the private cost of voting.

The expected benefits of voting depend on the type of voter (A or B) and the signal he or she has received (a or b). The benefits accrue only if an additional vote for the favored candidate affects the outcome of the election, that is, the voter is *pivotal*. Let $\Pr[Piv_A \mid \alpha]$ denote the probability that an additional vote for A is pivotal in state α. Denote by $\Pr[Piv_A \mid \beta]$, $\Pr[Piv_B \mid \alpha]$ and $\Pr[Piv_B \mid \beta]$ the other pivotal probabilities (the exact determination of these is given below). The private signal received by the voter determines the relative likelihoods of the two states. Thus, the expected benefit of voting to a voter of type A who receives a signal a is $q_a \Pr[Piv_A \mid \alpha] V + (1 - q_a) \Pr[Piv_A \mid \beta] v$. Such a voter participates only if his cost does not exceed the benefit. Thus, there is a cost threshold c_{Aa} that determines the participation decision. Similar considerations apply to other type-signal pairs.

The main result of the paper is that even though voting is solely on grounds of ideology, equilibrium participation rates ensure that in large elections, the outcome is socially optimal.[5] Formally,

[5] We calculate social welfare absent voting costs. This is without consequence because, in large elections, the per capita costs of voting go to zero.

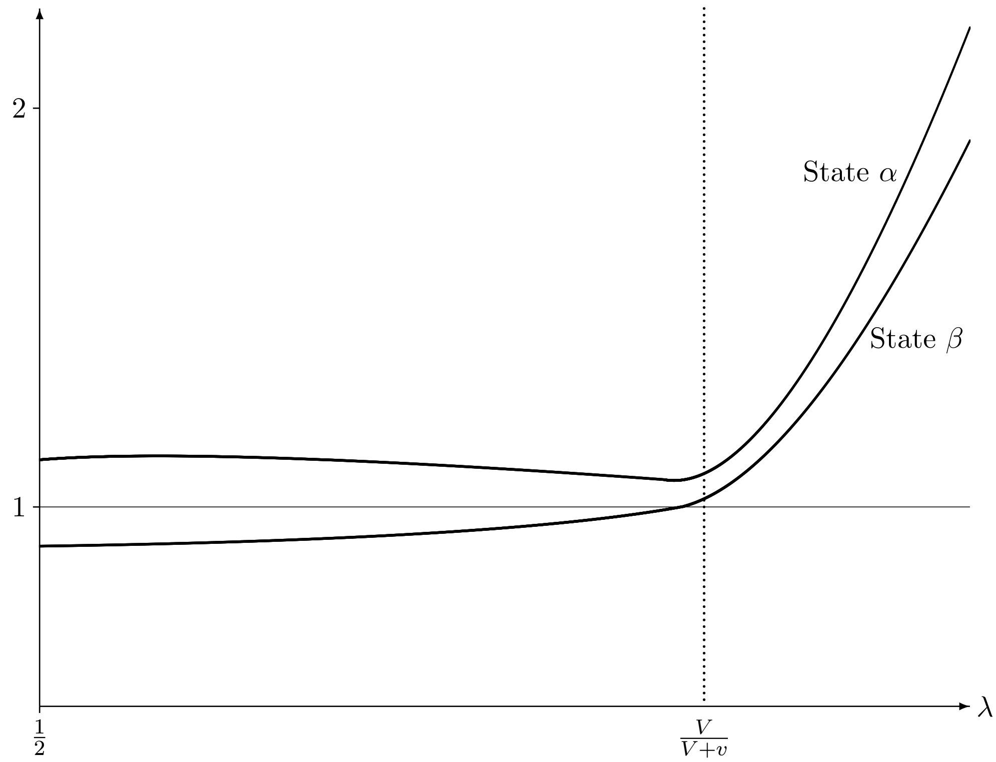

Figure 1: Ratio of A to B Votes

Theorem 1 *In large elections, voluntary voting is always welfare optimal. Precisely, in any equilibrium of a large election, if competence is efficient, A is elected in state* α *and B in state* β*; and (ii) if ideology is efficient, then A is elected in both states.*

The theorem may be illustrated by means of an example.

Example 4.1 *Suppose that* $\pi = \frac{1}{2}$ *and the signal precisions are* $\frac{2}{3}$ *in each state, so that* $q_a = q_b = \frac{2}{3}$*. Let* $V = 1.5$ *and* $v = 0.5$ *and suppose that voting costs are uniformly distributed. Notice that competence is efficient if the share of A voters,* $\lambda < \frac{V}{V+v} = \frac{3}{4}$ *and ideology is efficient if* $\lambda > \frac{3}{4}$*. Figure 4.1 depicts the equilibrium vote ratios in the two states as a function of* λ *when* $n = 10^6$*.*

The remainder of the paper is devoted to establishing the theorem. First, in Section 4.1 we show how the equilibrium participation rates are determined. In Section 4.2, we study the asymptotic behavior of the participation rates and expected turnout. We will show that while the participation rates go to zero as the expected electorate increases, the expected turnout increases without bound. Section 5 examines asymptotic vote share ratios in the special case in which costs are *uniformly* distributed and establishes the conclusion the theorem for this case. Finally, Section

6 shows that the results for the case of uniformly distributed costs to extend to the general model.

4.1 Equilibrium Conditions

As discussed above, under voluntary voting, equilibrium is characterized by the cost thresholds, c_{Aa}, c_{Ab}, c_{Ba}, c_{Bb}, of the conflicted and unconflicted voters of the two types. For each type-signal pair, these are determined by equating the benefits of voting to the costs. Thus,

$$\begin{aligned} c_{Aa} &= q_a \Pr[Piv_A \mid \alpha] V + (1-q_a) \Pr[Piv_A \mid \beta] v \\ c_{Ab} &= (1-q_b) \Pr[Piv_A \mid \alpha] V + q_b \Pr[Piv_A \mid \beta] v \\ c_{Ba} &= (1-q_a) \Pr[Piv_B \mid \beta] V + q_a \Pr[Piv_B \mid \alpha] v \\ c_{Bb} &= q_b \Pr[Piv_B \mid \beta] V + (1-q_b) \Pr[Piv_B \mid \alpha] v \end{aligned} \tag{7}$$

Equivalently, the equilibrium can be expressed in terms of participation rates, p_{Aa}, p_{Ab}, p_{Ba}, p_{Bb}, where $p_{Aa} = F(c_{Aa})$, $p_{Ba} = F(c_{Ba})$, etc. The participation rates in turn determine the pivot probabilities.

It remains to specify how the pivot probabilities are calculated. In the Poisson model, these depend only on the *expected* number of votes. In state α, the expected number of votes for A and B, denoted by σ_A and σ_B, respectively, are

$$\begin{aligned} \sigma_A &= n \times \lambda (\Pr[a \mid \alpha] p_{Aa} + \Pr[b \mid \alpha] p_{Ab}) \\ \sigma_B &= n \times (1-\lambda) (\Pr[a \mid \alpha] p_{Ba} + \Pr[b \mid \alpha] p_{Bb}) \end{aligned} \tag{8}$$

To see how σ_A is derived, for instance, notice that with probability λ a voter's ideology is A. With probability $\Pr[a \mid \alpha]$, this voter is unconflicted and turns out to vote at rate p_{Aa} while, with probability $\Pr[b \mid \alpha]$, this voter is conflicted and turns out to vote at rate p_{Ab}. The expected number of votes for B in state α, given by σ_B is calculated in similar fashion. Analogously, let τ_A and τ_B be the expected number of votes for A and B, respectively, in state β. We then have

$$\begin{aligned} \tau_A &= n \times \lambda (\Pr[a \mid \beta] p_{Aa} + \Pr[b \mid \beta] p_{Ab}) \\ \tau_B &= n \times (1-\lambda) (\Pr[a \mid \beta] p_{Ba} + \Pr[b \mid \beta] p_{Bb}) \end{aligned} \tag{9}$$

Since it may be possible for voters to abstain, it is only required that $\sigma_A + \sigma_B \leq n$ and $\tau_A + \tau_B \leq n$.

A vote is pivotal only if it either breaks a tie or it leads to a tie. The probability of a tie in state α is (see Myerson, 1998)

$$\Pr[T \mid \alpha] = e^{-\sigma_A - \sigma_B} \sum_{k=0}^{\infty} \frac{\sigma_A^k}{k!} \frac{\sigma_B^k}{k!} \tag{10}$$

while the probability that A falls one vote short in state α is

$$\Pr[T_{-1} \mid \alpha] = e^{-\sigma_A - \sigma_B} \sum_{k=1}^{\infty} \frac{\sigma_A^{k-1}}{(k-1)!} \frac{\sigma_B^k}{k!} \tag{11}$$

The probability $\Pr[T_{+1} \mid \alpha]$ that A is ahead by one vote may be written by exchanging σ_A and σ_B in (11). The corresponding probabilities in state β are obtained by substituting τ for σ.

Thus the probability that an additional vote for A is pivotal in state α is

$$\Pr[Piv_A \mid \alpha] = \tfrac{1}{2}\Pr[T \mid \alpha] + \tfrac{1}{2}\Pr[T_{-1} \mid \alpha]$$

and that an additional vote for B is pivotal in state α is

$$\Pr[Piv_B \mid \alpha] = \tfrac{1}{2}\Pr[T \mid \alpha] + \tfrac{1}{2}\Pr[T_{+1} \mid \alpha]$$

The pivot probabilities in state β are determined analogously.+

Since the pivot probabilities are continuous functions of the participation rates, by Brouwer's Theorem there exists a solution $(p_{Aa}, p_{Ab}, p_{Ba}, p_{Bb})$ to (7) which determines the equilibrium participation rates (and hence, also the corresponding equilibrium cost thresholds). It is easy to see that the solution must be interior, that is, $p_{Aa} \in (0, 1)$, etc. Thus,

Proposition 4.1 *Under voluntary voting, an equilibrium always exists. Furthermore, all equilibria entail positive participation rates for all type-signal pairs.*

4.2 Turnout in Large Elections

We will show that the expected turnouts for the four kinds of voters, $np_{Aa}, np_{Ab}, np_{Ba}$ and np_{Bb} all tend to infinity as n increases.

Our model shares the feature common to costly voting models that participation rates fall to zero as the size of the electorate increases. Were this not the case, say, for unconflicted A voters, then the probability that any voter is pivotal in any state would also fall to zero. But then the benefits of voting would also go to zero, contradicting the fact that unconflicted A voters continue to show up at positive rates. Thus,

Proposition 4.2 *As n goes to infinity, all turnout rates $p_{Aa}, p_{Ab}, p_{Ba}, p_{Bb}$ go to zero.*

While turnout rates go to zero, the behavior of the *expected* turnout is determined by speed with which these rates converges to zero. Roughly, if the rate of convergence were n or faster, then the expected turnout would be bounded. This, however, can be ruled out. If the expected turnout were bounded, then voters would be pivotal even in the limit and so would have a positive incentive to participate, contradicting the proposition above. Thus, at least for at least *one* type-signal pair, the expected turnout must go to infinity. It can be shown, however, that all expected turnouts move together (their ratios are bounded), so that, in fact, the expected turnout for *every* type-signal pair goes to infinity. Formally,

Proposition 4.3 *As n goes to infinity, all expected turnouts $np_{Aa}, np_{Ab}, np_{Ba}, np_{Bb}$ also go to infinity.*

Proposition 4.3 is essential to establishing the welfare optimality of large elections under voluntary voting. It guarantees that if the vote shares favor the "correct" candidate, then that candidate wins with probability close to one.

5 Vote Shares: Uniform Distribution

Under the assumption that the distribution of costs is *uniform* on $[0,1]$ and $V \leq 1$, the equilibrium conditions in (7) become (since $c_{Aa} = p_{Aa}$ etc.):

$$\begin{aligned}
p_{Aa} &= q_a \Pr[Piv_A \mid \alpha] V + (1-q_a) \Pr[Piv_A \mid \beta] v \\
p_{Ab} &= (1-q_b) \Pr[Piv_A \mid \alpha] V + q_b \Pr[Piv_A \mid \beta] v \\
p_{Ba} &= (1-q_a) \Pr[Piv_B \mid \beta] V + q_a \Pr[Piv_B \mid \alpha] v \\
p_{Bb} &= q_b \Pr[Piv_B \mid \beta] V + (1-q_b) \Pr[Piv_B \mid \alpha] v
\end{aligned}$$

Using the definitions in (8) and (9), we obtain that the expected number of votes in the two states are

$$\begin{aligned}
\sigma_A &= n\lambda (r \Pr[Piv_A \mid \alpha] V + (1-r) \Pr[Piv_A \mid \beta] v) \\
\sigma_B &= n(1-\lambda)((1-r) \Pr[Piv_B \mid \beta] V + r \Pr[Piv_B \mid \alpha] v) \\
\tau_A &= n\lambda ((1-s) \Pr[Piv_A \mid \alpha] V + s \Pr[Piv_A \mid \beta] v) \\
\tau_B &= n(1-\lambda)(s \Pr[Piv_B \mid \beta] V + (1-s) \Pr[Piv_B \mid \alpha] v)
\end{aligned} \tag{12}$$

where r and s are the preposteriors in state α and β, as defined in (5).

Our first result rules out the perverse case where the *less* competent candidate gets majority vote share in each state. It shows that if B has the majority in state α, then B also has the majority in state β and, moreover, with a bigger expected vote differential.

Lemma 5.1 *If $\sigma_A \leq \sigma_B$, then $\tau_B - \tau_A > \sigma_B - \sigma_A$.*

Proof. Recall that if $\sigma_A \leq \sigma_B$, then $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \alpha]$ (see Lemma A.1) Thus

$$\frac{s\sigma_B - (1-r)\tau_B}{s\sigma_A - (1-r)\tau_A} = \frac{1-\lambda}{\lambda} \frac{v}{V} \frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_A \mid \alpha]} < 1$$

since $\lambda > \frac{1}{2}$ and $V > v$. Thus, $s\sigma_A - (1-r)\tau_A > s\sigma_B - (1-r)\tau_B$ or equivalently, $(1-r)(\tau_B - \tau_A) > s(\sigma_B - \sigma_A)$. Since $1 - r < s$, the result follows. ■

5.1 Ideology is Efficient

We first consider the relatively simpler case when ideology is efficient; that is, it is socially optimal to elect A in both states. This occurs when the share of A types is large relative to the losses associated with electing A in state β. Precisely, this occurs when $\lambda v \geq (1-\lambda) V$.

It is intuitive that in state α, where A is both competent and heavily favored on ideological grounds, that A should obtain a majority. The next lemma verifies this intuition for the case of uniformly distributed costs.

Lemma 5.2 *If ideology is efficient, then $\sigma_A > \sigma_B$.*

Proof. Suppose to the contrary that $\sigma_A \leq \sigma_B$. From Lemma 5.1 we know that $\tau_A < \tau_B$. Together, these imply that $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \alpha]$ and $\Pr[Piv_A \mid \beta] > \Pr[Piv_B \mid \beta]$.

Since $\lambda v \geq (1-\lambda) V$,

$$\begin{aligned}
\sigma_A &= n\lambda \left(r \Pr[Piv_A \mid \alpha] V + (1-r) \Pr[Piv_A \mid \beta] v\right) \\
&> n(1-\lambda)\left(r \Pr[Piv_B \mid \alpha] v + (1-r) \Pr[Piv_B \mid \beta] V\right) \\
&= \sigma_B
\end{aligned}$$

which is a contradiction. ■

The situation is a little delicate in state β since there is a tension between competence and ideology. But when $\lambda v \geq (1-\lambda) V$, candidate A is so heavily favored on ideological grounds that A indeed gets a majority in state β also.

Lemma 5.3 *If ideology is efficient, then $\tau_A > \tau_B$.*

Proof. Suppose to the contrary that $\tau_A \leq \tau_B$. We know from Lemma 5.2 that $\sigma_A > \sigma_B$. Then

$$r(\tau_B - \tau_A) - (1-s)(\sigma_B - \sigma_A) > 0$$

But since $\tau_A \leq \tau_B$, $\Pr[Piv_A \mid \beta] \geq \Pr[Piv_B \mid \beta]$ and combining this with the fact that $\lambda v \geq (1-\lambda) V$ implies that

$$\frac{r\tau_B - (1-s)\sigma_B}{r\tau_A - (1-s)\sigma_A} = \frac{1-\lambda}{\lambda}\frac{V}{v}\frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_A \mid \beta]} \leq 1$$

and this contradicts the inequality above. ■

Thus, we conclude that no matter what the size of the electorate, A has a majority in both states when ideology is efficient. In large elections, this implies that A is elected with probability close to one.

Proposition 5.1 *If ideology is efficient, then A has a majority in both states.*

5.2 Competence is Efficient

The argument that voluntary voting leads to socially optimal outcomes in large elections when competence is efficient is somewhat involved. In fact, when competence is efficient even the argument that A wins a majority in state α is not straightforward. To get a feel for some of the issues involved in showing that $\sigma_A > \sigma_B$, consider a direct comparison of the two. Suppose to contrary that B gets a majority in state α, that is, $\sigma_A \leq \sigma_B$. Since the "wrong" candidate cannot get a majority in both states (Lemma 5.1) it must be that B has a majority in state β also; that is, $\tau_A < \tau_B$. Together, these imply that $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \alpha]$ and $\Pr[Piv_A \mid \beta] > \Pr[Piv_B \mid \beta]$. Now a direct comparison of the vote shares in state α:

$$\begin{aligned}
\sigma_A &= n\lambda \left(r \Pr[Piv_A \mid \alpha] V + (1-r) \Pr[Piv_A \mid \beta] v\right) \\
\sigma_B &= n(1-\lambda)\left(r \Pr[Piv_B \mid \alpha] v + (1-r) \Pr[Piv_B \mid \beta] V\right)
\end{aligned}$$

shows that while the first terms in each may be readily compared, a comparison of the second terms is ambiguous. In fact, since $\lambda v < (1-\lambda) V$ it could well be the case that the ranking of the pivot probabilities is not enough to guarantee that $\lambda v \Pr[Piv_A \mid \beta] > (1-\lambda) V \Pr[Piv_B \mid \beta]$. Hence, unlike the case when ideology is efficient, a term-by-term comparison is not definitive.

The next lemma establishes the intuitive property that if an A vote is more likely to be pivotal than a B vote is in state β, then A voters have a greater incentive to show up and so A gets a majority vote share.

Lemma 5.4 *If* $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \beta]$, *then* $\sigma_A > \sigma_B$.

Proof. Suppose to the contrary that $\sigma_A \leq \sigma_B$. Then we know from Lemma 5.1 that $\tau_A < \tau_B$. Thus, $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \alpha]$ and $\Pr[Piv_A \mid \beta] > \Pr[Piv_B \mid \beta]$. Since $\lambda \geq \frac{1}{2}$ and the preposterior $r > \frac{1}{2}$,

$$
\begin{aligned}
\sigma_A &= n\lambda \left(r \Pr[Piv_A \mid \alpha] V + (1-r) \Pr[Piv_A \mid \beta] v\right) \\
&\geq n(1-\lambda)\left(r \Pr[Piv_A \mid \alpha] V + (1-r) \Pr[Piv_A \mid \beta] v\right) \\
&= n(1-\lambda)\left(r \Pr[Piv_A \mid \alpha] v + (1-r) \Pr[Piv_A \mid \beta] v + r \Pr[Piv_A \mid \alpha](V-v)\right) \\
&> n(1-\lambda)\left(r \Pr[Piv_B \mid \alpha] v + (1-r) \Pr[Piv_B \mid \beta] v + r \Pr[Piv_A \mid \alpha](V-v)\right) \\
&= n(1-\lambda)\left((1-r) \Pr[Piv_B \mid \beta] V + r \Pr[Piv_B \mid \alpha] v\right) \\
&\quad + n(1-\lambda)\left(r \Pr[Piv_A \mid \alpha] - (1-r) \Pr[Piv_B \mid \beta]\right)(V-v) \\
&\geq \sigma_B
\end{aligned}
$$

■

In Appendix C we then show that all vote share configurations where B gets a majority vote share in state α imply that the election is closer in state α than in state β. This in turn implies that the pivotality condition in Lemma 5.4 holds. Thus, we obtain

Proposition 5.2 *A has a majority vote share in state α; that is, $\sigma_A > \sigma_B$.*

We are ready to turn to the more delicate issue of whether B enjoys a majority vote share in state β. To establish this, we first note that if A enjoys a majority in both states then he must win with a greater majority in state α (Lemmas D.1 and D.2 in Appendix D). Now as the size of the electorate increases, this implies the pivotal vote is increasingly likely to occur in state β than in state α. Thus, turnout incentives for B types are strengthened while those for A types are weakened and this would imply that B enjoys a majority voter share in both states, which is impossible (Lemma D.6). The details are in Appendix D.

Proposition 5.3 *Suppose competence is efficient. In large elections, B has a majority vote share in state β; that is, $\tau_A < \tau_B$.*

To summarize, we have thus shown that when voting costs are uniformly distributed, the conclusion of Theorem 1 holds. Before extending the proof to arbitrary cost distributions, we discuss two issues.

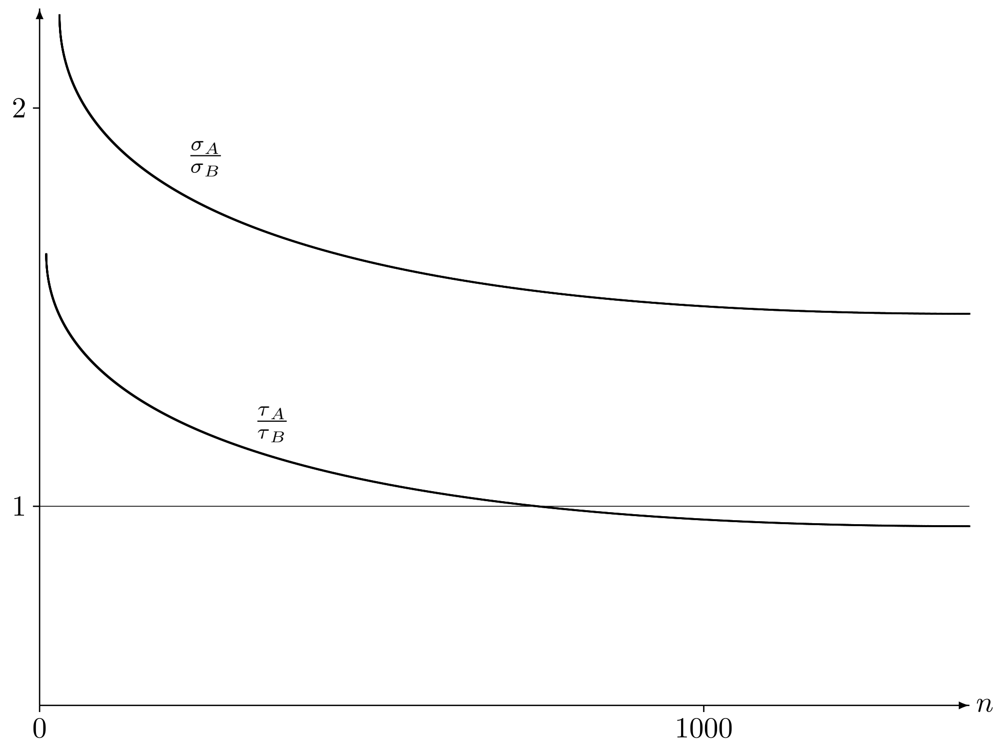

Figure 2: Equilibrium Vote Ratios

5.3 The Role of Large Numbers

The main result of this paper relies on there being a large number of voters. The "large n" assumption plays two roles. The first is just to invoke the law of large numbers—if the ratio of the expected number of votes favors a particular candidate, then in large elections that candidate is elected with probability close to one. The second effect is more subtle—it could be that when n is small, the ratio of the expected number of votes favors the "wrong" candidate but as we have shown above, the ratio always favors the "correct" candidate in large elections.

Example 5.1 *Suppose that the states are equally likely ($\pi = \frac{1}{2}$) and the signal precisions are $\frac{2}{3}$ in each state, so that $q_a = q_b = \frac{2}{3}$. Let $V = 1$ and $v = 0$ and suppose that costs are uniformly distributed. Finally, suppose that the share of A voters, $\lambda = \frac{3}{4}$. Figure 5.1 depicts the two vote ratios as a function of n.*

Since $v = 0$, it is the case that for all λ, competence is efficient—it is welfare maximizing to elect A in state α and B in state β. As depicted in Figure 5.1, the vote ratios favor A in α and B in β only when n is large enough. To see why large n matters, recall that when $v = 0$, the equilibrium conditions in (12) imply that the vote ratio in state β is

$$\frac{\tau_A}{\tau_B} = \frac{\lambda (1-s)}{(1-\lambda) s} \frac{\Pr [Piv_A \mid \alpha]}{\Pr [Piv_B \mid \beta]}$$

The first part of this expression is a comparison of signal precision with ideological bias. The second term is a ratio of pivot probabilities. As the example demonstrates, when λ is high relative to s, the first term is greater than one and as a result, it could be that $\tau_A > \tau_B$. As argued above, the ratio of the pivot probabilities falls as n increases—a pivotal vote is increasingly likely to occur in state β than in state α. Eventually, it falls sufficiently low so as to overwhelm the ideological bias and, as a consequence, B starts to enjoy a majority vote in state β for n sufficiently large.

To see the second effect, consider the limiting case when $n \to 0$. In that case, a voter of either type is almost surely pivotal. Thus pivotality does not *differentially* affect the decision of the two types of whether to vote or not—A and B types participate at the same rate. Now the model effectively becomes one with compulsory voting and as we argued above, compulsory voting does not lead to efficient outcomes. Thus large numbers are essential to establishing the main result of this paper.

5.4 Conflicted versus Unconflicted Voters

One may surmise that when competence is efficient, unconflicted voters turn out in larger numbers relative to conflicted voters leading to the result that the right candidate is elected in each state. Underlying this intuition is the simple observation that the "value of voting" is higher for unconflicted types than it is for conflicted. This, however, ignores differences in the likelihoods of being pivotal in the two states. As the following example shows, the latter effect can dominate, leading conflicted types to show up *more* often than their unconflicted counterparts. Despite this, the right candidate always wins. Figure 3 depicts the ratio of unconflicted to conflicted voters for both types for the parameters in Example 4.1. Notice that while unconflicted type B voters participate at higher rates than conflicted type B voters, this is not true of type A voters.

6 General Cost Distributions

In the previous section, we established the main result for the case when the distribution of voting costs was uniform. In this case, the expected number of votes in a state could be written as linear functions of the pivotal probabilities (see 12). Now suppose that all voters draw costs from some arbitrary distribution F. The ratio of the expected number of votes in state α are now

$$\frac{\sigma_A}{\sigma_B} = \frac{\lambda}{1-\lambda} \frac{\Pr [a \mid \alpha] F (c_{Aa}) + \Pr [b \mid \alpha] F (c_{Ab})}{\Pr [a \mid \alpha] F (c_{Ba}) + \Pr [b \mid \alpha] F (c_{Bb})}$$

Recall that as $n \to \infty$, all cost thresholds go to zero and at the same rate. Thus, for large n, we can write

$$\frac{\sigma_A}{\sigma_B} \approx \frac{\lambda}{1-\lambda} \frac{\Pr [a \mid \alpha] F' (0) c_{Aa} + \Pr [b \mid \alpha] F' (0) c_{Ab}}{\Pr [a \mid \alpha] F' (0) c_{Ba} + \Pr [b \mid \alpha] F' (0) c_{Bb}}$$

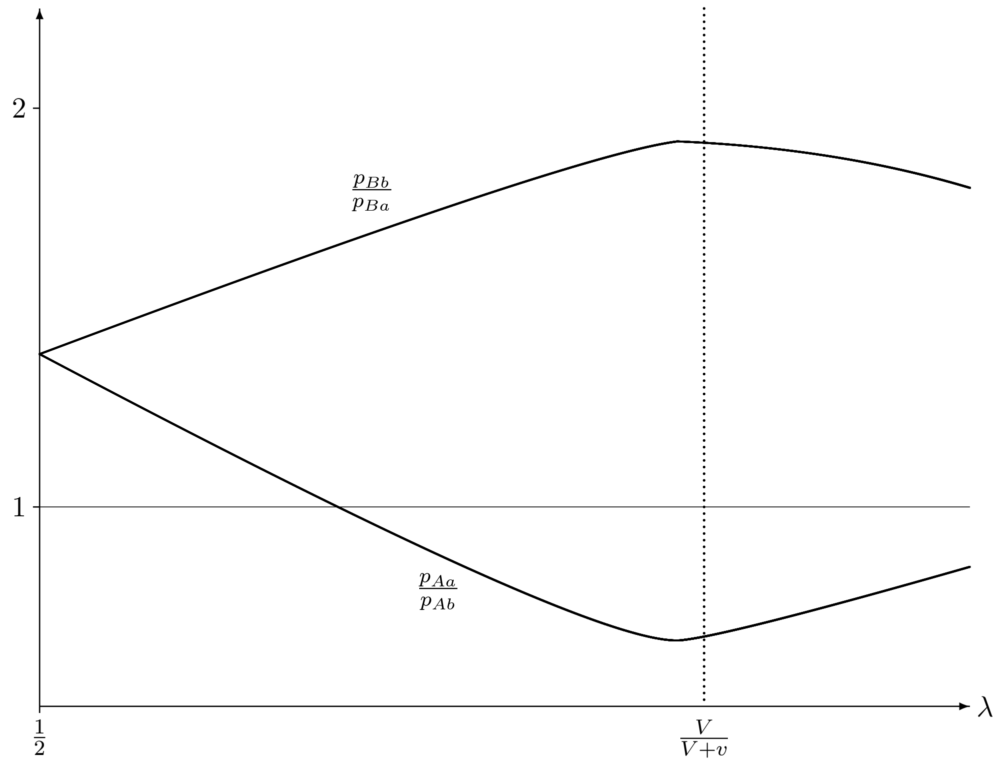

Figure 3: Ratio of Unconflicted to Conflicted Voters

and since $F'(0) > 0$, this reduces to the expression above. Thus, for large n, the ratio of the expected number of votes is the same as in (12). A similar expression holds in state β.

7 Discussion

The main result of this paper relies essentially on the assumption that the lower support of the cost distribution is 0. If instead, all voters had voting costs of at least $\varepsilon > 0$ then large elections suffer from the familiar problem (Downs, 1957) that turnout is bounded in the limit and, as a consequence, information does not aggregate regardless of preferences over ideology versus competence. In principle, this situation may be remedied by introducing a small fine, equal to ε for not voting, and redistributing the proceeds in lump-sum fashion. This effectively shifts the cost distribution ε to the left and the results of our model apply.

Costly voting models suffer from the problem that, in large elections, turnout rates go to zero, which is clearly at odds with reality. One "solution" to this difficulty is to introduce the notion of voting as "duty" (i.e., voters obtain positive utility from the act of voting). See, for example, Riker and Ordeshook (1968). If this duty term is large enough so that a positive fraction of voters derive a net positive utility from voting, then, in large elections, these would be the only voters coming to the polls. In our model, these voters vote purely ideologically and, since turnout is no longer sensitive to the trade-off between ideology and competence, the election will always be decided in favor of the candidate whose ideology coincides with the majority. In this case, the problem is too much turnout rather than to little. The policy solution is to impose a tax so as to dampen the enthusiasm for these voters to come to the polls.

We adopt the Poisson model introduced by Myerson (1998 & 2000). This model has the arguably realistic feature that the exact size of the electorate is random and unknown to voters. More importantly, it considerably simplifies the analysis—especially, the computation of pivot probabilities—compared to a model with a fixed and commonly known number of voters. This modeling convenience is of no consequence in large elections. Indeed, Myerson (1998) has demonstrated, the qualitative predictions of Poisson voting models, especially when the (expected) size of the electorate is large, are identical to those with a fixed electorate.

While our research concerns majoritarian elections which constitute the bulk of elections in first-past-the-post systems such as in the US, it is an open question whether our conclusions extend to supermajority rules. The key hurdle is technical—the set of pivotal events change depending on the voting rule. Examining the statistical properties of these events requires different tools than those we used for majority rule. This remains for future research.

8 Conclusion

When ideology is important, and perhaps dominates voters' choices at the polls, it is not surprising that the information aggregation properties of elections suffer. In effect, ideology blocks out all other information. What is needed is some channel besides voting itself for conveying this information. In this paper, we identify a natural and arguably realistic additional channel—voters' feet. That is, when voters are allowed to express preferences with turnout, large majoritarian elections continue to perform admirably: the Condorcet Jury Theorem is restored even when voting is based purely on ideology.

An implication of our finding concerns "compulsory voting" policies whose intent is to induce full voter participation. While these policies are designed to ensure that the preferences of all voters are reflected at the polls, in our model, they have the perverse effect of creating a tyranny of the majority, much to the detriment of social welfare. Rather than encouraging the thoughtful exercise of democracy, such policies reduce the informativeness of elections and merely ensure that the majority ideology gets its preferred candidate, regardless of competence—even when all voters are well-informed and rational.

This is not to say that all schemes encouraging voter participation are misguided. Our main result relies crucially on the assumption that a fraction of voters have negligible voting costs. As discussed above, if all voters have costs above some positive threshold then the optimal policy would be to impose a modest fine for not voting, equal to the threshold. Redistributing the proceeds of the fine in lump-sum fashion restores the conditions needed for our main result and again leads to first-best outcomes. Viewed in this light, "compulsory" voting schemes that impose mild fines, as in countries like Australia and Belgium, are more likely to produce desirable outcomes than more drastic sanctions, as in Bolivia.

The increasing importance of ideology has been informally cited as a worrying trend in US politics, leading to polarization. One might expect this trend to produce elected officials who are distinguished for their ideological purity and little else. Our result suggests that this conclusion is too pessimistic. While ideology may well dominate other considerations at the polls, competence (or valence) considerations continue to drive turnout and, as we have shown, this force is sufficient to trump ideology whenever it is socially efficient to do so.

A Pivot Probabilities

In this appendix, we collect some useful facts about the pivot probabilities in Poisson voting games.

In what follows, it will be useful to rewrite the pivot probabilities in terms of the *modified Bessel functions* (see Abramowitz and Stegun, 1965),

$$I_0\left(z\right)=\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^k}{k!}\frac{\left(\frac{z}{2}\right)^k}{k!} \text{ and } I_1\left(z\right)=\sum_{k=1}^{\infty}\frac{\left(\frac{z}{2}\right)^{k-1}}{(k-1)!}\frac{\left(\frac{z}{2}\right)^k}{k!}$$

Using these, we can rewrite the probabilities associated with close elections as

$$\begin{aligned}
\Pr\left[T\mid\alpha\right] &= e^{-\sigma_A-\sigma_B}I_0\left(2\sqrt{\sigma_A\sigma_B}\right)\\
\Pr\left[T_{-1}\mid\alpha\right] &= e^{-\sigma_A-\sigma_B}I_1\left(2\sqrt{\sigma_A\sigma_B}\right)\sqrt{\tfrac{\sigma_B}{\sigma_A}}\\
\Pr\left[T_{+1}\mid\alpha\right] &= e^{-\sigma_A-\sigma_B}I_1\left(2\sqrt{\sigma_A\sigma_B}\right)\sqrt{\tfrac{\sigma_A}{\sigma_B}}
\end{aligned}$$

Again, the corresponding probabilities in state β are found by substituting τ for σ.

The following result is used repeatedly. It says that a vote in favor of the losing side (with the smaller expected vote total) is more likely to be pivotal than a vote in favor of the winning side.

Lemma A.1 *$\sigma_A \leq \sigma_B$ if and only if $\Pr\left[Piv_A\mid\alpha\right]\geq\Pr\left[Piv_B\mid\alpha\right]$. Similarly, $\tau_A\leq\tau_B$ if and only if $\Pr\left[Piv_A\mid\beta\right]\geq\Pr\left[Piv_B\mid\beta\right]$.*

Proof. Since

$$\begin{aligned}
\Pr\left[Piv_A\mid\alpha\right]-\Pr\left[Piv_B\mid\alpha\right] &= \tfrac{1}{2}\left(\Pr\left[T_{-1}\mid\alpha\right]-\Pr\left[T_{+1}\mid\alpha\right]\right)\\
&= \tfrac{1}{2}e^{-\sigma_A-\sigma_B}I_1\left(2\sqrt{\sigma_A\sigma_B}\right)\left(\sqrt{\tfrac{\sigma_B}{\sigma_A}}-\sqrt{\tfrac{\sigma_A}{\sigma_B}}\right)
\end{aligned}$$

and the result is immediate.

The proof for state β is analogous. ■

For our asymptotic results it is useful to note that when z is large, the modified Bessel functions can be approximated as follows[6] (see Abramowitz and Stegun, 1965, p. 377)

$$I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}} \approx I_1(z)$$

This implies, for instance, that

$$\begin{aligned}\Pr[Piv_A \mid \alpha] &\approx e^{-\sigma_A - \sigma_B} e^{2\sqrt{\sigma_A \sigma_B}} \frac{1}{\sqrt{4\pi\sqrt{\sigma_A\sigma_B}}} \left(1 + \sqrt{\frac{\sigma_B}{\sigma_A}}\right) \\ &= e^{-\left(\sqrt{\sigma_A} - \sqrt{\sigma_B}\right)^2} \frac{1}{\sqrt{4\pi\sqrt{\sigma_A\sigma_B}}} \left(1 + \sqrt{\frac{\sigma_B}{\sigma_A}}\right) \qquad (13)\end{aligned}$$

Similar expressions obtain for the three other pivot probabilities.

Using this approximation we obtain

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_B \mid \alpha]} \approx \frac{\left(1 + \sqrt{\frac{\sigma_B}{\sigma_A}}\right)}{\left(1 + \sqrt{\frac{\sigma_A}{\sigma_B}}\right)} = \sqrt{\frac{\sigma_B}{\sigma_A}} \qquad (14)$$

B Proof of Proposition 4.3

The proof consists of a series of lemmas. The first lemma establishes that as n increases, the ratio of cost thresholds for conflicted and unconflicted voters of a given type are bounded.

Lemma B.1 *Along any sequence of equilibria,*

$$\frac{1-q_a}{q_b} \leq \liminf \frac{c_{Aa}}{c_{Ab}} \leq \limsup \frac{c_{Aa}}{c_{Ab}} \leq \frac{q_a}{1-q_b}$$

and

$$\frac{1-q_b}{q_a} \leq \liminf \frac{c_{Bb}}{c_{Ba}} \leq \limsup \frac{c_{Bb}}{c_{Ba}} \leq \frac{q_b}{1-q_a}$$

Proof. First, suppose that there is a subsequence of n's (and an equilibrium for each n) such that along this subsequence

$$\lim \frac{c_{Aa}}{c_{Ab}} < \frac{1-q_a}{q_b}$$

From the threshold equations (7), we can write: for large n along the subsequence

$$\frac{c_{Aa}}{c_{Ab}} = \frac{q_a \Pr[Piv_A \mid \alpha] V + (1-q_a) \Pr[Piv_A \mid \beta] v}{(1-q_b) \Pr[Piv_A \mid \alpha] V + q_b \Pr[Piv_A \mid \beta] v} < \frac{1-q_a}{q_b}$$

By cross-multiplying, this is easily seen to be equivalent to $q_a + q_b < 1$, which contradicts (4).

[6] $X(n) \approx Y(n)$ means that $\lim_{n\to\infty} (X(n)/Y(n)) = 1$.

Similarly, suppose that there is a subsequence along which

$$\lim \frac{c_{Aa}}{c_{Ab}} > \frac{q_a}{1 - q_b}$$

In that case, for large n,

$$\frac{c_{Aa}}{c_{Ab}} = \frac{q_a \Pr[Piv_A \mid \alpha] V + (1 - q_a) \Pr[Piv_A \mid \beta] v}{(1 - q_b) \Pr[Piv_A \mid \alpha] V + q_b \Pr[Piv_A \mid \beta] v} > \frac{q_a}{1 - q_b}$$

which again contradicts (4).

The proofs for the other inequalities are similar. ■

The next lemma shows that the ratio of the participation rates of conflicted and unconflicted voters of a given type also stay bounded.

Lemma B.2 *Along any sequence of equilibria,*

$$\begin{aligned} 0 &< \liminf \frac{p_{Aa}}{p_{Ab}} \leq \limsup \frac{p_{Aa}}{p_{Ab}} < \infty \\ 0 &< \liminf \frac{p_{Bb}}{p_{Ba}} \leq \limsup \frac{p_{Bb}}{p_{Ba}} < \infty \end{aligned}$$

Proof. Suppose to the contrary that along some subsequence

$$\lim \frac{p_{Aa}}{p_{Ab}} = 0$$

This is the same as saying that along this subsequence, for all ε, there exists an N such that for all $n > N$,

$$\frac{p_{Aa}}{p_{Ab}} < \varepsilon$$

As above, let $c_{Aa} = F^{-1}(p_{Aa})$ and $c_{Ab} = F^{-1}(p_{Ab})$ be the corresponding sequence of cost thresholds. Then we have that for large n along this sequence,

$$\frac{F(c_{Aa})}{F(c_{Ab})} < \varepsilon$$

which is equivalent to

$$\frac{c_{Aa}}{c_{Ab}} < \frac{F^{-1}(\varepsilon F(c_{Ab}))}{c_{Ab}}$$

Now by Lemma B.1, we have that for large n,

$$\frac{1 - q_a}{q_b} < \frac{F^{-1}(\varepsilon F(c_{Ab}))}{c_{Ab}}$$

or equivalently, that

$$\frac{F\left(\frac{1-q_a}{q_b} c_{Ab}\right)}{F(c_{Ab})} < \varepsilon$$

and so along this subsequence we have

$$\lim \frac{F\left(\frac{1-q_a}{q_b} c_{Ab}\right)}{F\left(c_{Ab}\right)} = 0$$

But since $\lim c_{Ab} = 0$, this contradicts the assumption that F has a positive density at 0,

$$\lim_{c_{Ab} \to 0} \frac{F\left(\frac{1-q_a}{q_b} c_{Ab}\right)}{F\left(c_{Ab}\right)} = \frac{F'(0)}{F'(0)} \left(\frac{1-q_a}{q_b}\right) > 0$$

On the other hand, suppose that along some subsequence

$$\lim \frac{p_{Ab}}{p_{Aa}} = 0$$

The same reasoning as above shows that along this subsequence

$$\lim \frac{F\left(\frac{1-q_a}{q_b} c_{Aa}\right)}{F\left(c_{Aa}\right)} = 0$$

which again contradicts our assumption about F.

The proof of the second set of inequalities is analogous. ■

Our next result is that either A or B type votes turn out in large numbers.

Lemma B.3 *Either* $\liminf np_{Aa} = \infty$ *or* $\liminf np_{Bb} = \infty$.

Proof. Suppose that for some subsequence, $\lim np_{Aa} < \infty$ and $\lim np_{Bb} < \infty$. Then from Lemma B.2 there is a further subsequence along which $\lim np_{Ab} < \infty$ and $\lim np_{Bb} < \infty$ also. This means that along this subsequence, the probability of being pivotal is not zero, contradicting the fact that all participation rates go to zero (Proposition 4.2). ■

Finally, we show that the ratio of unconflicted A voters to unconflicted B voters is bounded.

Lemma B.4 *Along any sequence of equilibria,* $0 < \liminf \frac{p_{Aa}}{p_{Bb}} \leq \limsup \frac{p_{Aa}}{p_{Bb}} < \infty$.

Proof. Suppose $\liminf \frac{p_{Aa}}{p_{Bb}} = 0$. Then from Lemma B.3 it must be that $\liminf np_{Bb} = \infty$ and so by Lemma B.2 $\liminf np_{Ba} = \infty$ also. It then also follows that for large n, $p_{Bb} > p_{Aa}$ (and hence $c_{Bb} > c_{Aa}$ also).

We will to argue that

$$\frac{\Pr\left[Piv_A \mid \alpha\right]}{\Pr\left[Piv_B \mid \alpha\right]} \to \infty \text{ and } \frac{\Pr\left[Piv_A \mid \beta\right]}{\Pr\left[Piv_B \mid \beta\right]} \to \infty$$

Note that since

$$\begin{aligned}
\Pr\left[Piv_A \mid \alpha\right] &= \tfrac{1}{2} \Pr\left[T \mid \alpha\right] + \tfrac{1}{2} \Pr\left[T_{-1} \mid \alpha\right] \\
\Pr\left[Piv_B \mid \alpha\right] &= \tfrac{1}{2} \Pr\left[T \mid \alpha\right] + \tfrac{1}{2} \Pr\left[T_{+1} \mid \alpha\right]
\end{aligned}$$

it is sufficient to show that

$$\frac{\Pr[T_{-1} \mid \alpha]}{\Pr[T_{+1} \mid \alpha]} \to \infty$$

But from (11) and the definitions of σ_A and σ_B,

$$\begin{aligned}
\frac{\Pr[T_{-1} \mid \alpha]}{\Pr[T_{+1} \mid \alpha]} &= \frac{\sigma_B}{\sigma_A} \\
&= \frac{n(1-\lambda)(\Pr[a \mid \alpha] p_{Ba} + \Pr[b \mid \alpha] p_{Bb})}{n\lambda(\Pr[a \mid \alpha] p_{Aa} + \Pr[b \mid \alpha] p_{Ab})} \\
&= \frac{(1-\lambda)\left(\Pr[a \mid \alpha] \frac{p_{Ba}}{p_{Bb}} + \Pr[b \mid \alpha]\right)}{\lambda\left(\Pr[a \mid \alpha] \frac{p_{Aa}}{p_{Bb}} + \Pr[b \mid \alpha] \frac{p_{Ab}}{p_{Aa}} \frac{p_{Aa}}{p_{Bb}}\right)}
\end{aligned}$$

and since $\frac{p_{Aa}}{p_{Bb}} \to 0$ while $\frac{p_{Ab}}{p_{Aa}}$ and $\frac{p_{Ba}}{p_{Bb}}$ are bounded (see Lemma B.2), we conclude that the ratio goes to infinity. The same argument holds in state β also. Thus, we have argued that

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_B \mid \alpha]} \to \infty \text{ and } \frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_B \mid \beta]} \to \infty$$

This in turn implies that

$$\frac{c_{Aa}}{c_{Bb}} = \frac{q_a \Pr[Piv_A \mid \alpha] V + (1-q_a)\Pr[Piv_A \mid \beta] v}{q_b \Pr[Piv_B \mid \beta] V + (1-q_b)\Pr[Piv_B \mid \alpha] v} \to \infty$$

which contradicts the fact that for large n, $c_{Aa} < c_{Bb}$.

The same argument applies if $\liminf \frac{p_{Bb}}{p_{Aa}} = 0$. ■

Lemmas B.3 and B.4 together complete the proof of Proposition 4.3.

C Proof of Proposition 5.2

Note All the results in this appendix are derived under the assumption that the distribution of voting costs is uniform.

Lemma C.1 *If $\sigma_A \leq \sigma_B$, then $\frac{\sigma_A}{\sigma_B} > \frac{\tau_A}{\tau_B}$.*

Proof. As in the proof of Lemma 5.1, $\sigma_A \leq \sigma_B$ implies that $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \alpha]$ and so

$$\frac{s\sigma_A - (1-r)\tau_A}{s\sigma_B - (1-r)\tau_B} = \frac{\lambda}{1-\lambda} \frac{V}{v} \frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_B \mid \alpha]} > 1$$

Lemma 5.1 also implies that $\tau_A < \tau_B$. Thus, we have

$$\frac{s\sigma_A - (1-r)\tau_A}{s\sigma_B - (1-r)\tau_B} > \frac{\tau_A}{\tau_B}$$

and the required inequality follows immediately (recall that $s\sigma_B - (1-r)\tau_B > 0$). ■

Lemma C.2 *If $\sigma_B \geq \tau_B$, then $\sigma_A > \sigma_B$.*

Proof. If $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \beta]$, then the conclusion follows from Lemma 5.4.

If $\Pr[Piv_A \mid \alpha] < \Pr[Piv_B \mid \beta]$, then suppose to the contrary that $\sigma_A \leq \sigma_B$. Thus, $\Pr[Piv_B \mid \alpha] \leq \Pr[Piv_A \mid \alpha]$ and so $\Pr[Piv_B \mid \alpha] < \Pr[Piv_B \mid \beta]$. But this implies that

$$\sigma_B - \tau_B = n(1-\lambda)(r+s-1)(\Pr[Piv_B \mid \alpha]v - \Pr[Piv_B \mid \beta]V) < 0$$

which is a contradiction. ■

Lemma C.3 *If $\sigma_B < \tau_B$ and $\sigma_A \geq \tau_A$, then $\sigma_A > \sigma_B$.*

Proof. Suppose to the contrary that $\sigma_A \leq \sigma_B$. Then we know from Lemma 5.1 that $\tau_A < \tau_B$.

Consider the function

$$g(x,y) = \tfrac{1}{2}e^{-x-y}\left(I_0(2\sqrt{xy}) + I_1(2\sqrt{xy})\sqrt{\frac{x}{y}}\right)$$

Notice that $\Pr[Piv_B \mid \alpha] = g(\sigma_A, \sigma_B)$ and $\Pr[Piv_B \mid \beta] = g(\tau_A, \tau_B)$. It is easily verified that

$$\begin{aligned} g_x(x,y) &= \frac{e^{-x-y}}{2\sqrt{xy}}(y-x)I_1(2\sqrt{xy}) > 0 \\ g_y(x,y) &= -\frac{1}{y}\frac{e^{-x-y}}{2\sqrt{xy}}\left(xI_1(2\sqrt{xy}) + (y-x)\sqrt{xy}I_0(2\sqrt{xy})\right) < 0 \end{aligned}$$

whenever $y > x$.

Now since $\sigma_B < \tau_B$ and $\sigma_A \geq \tau_A$,

$$\Pr[Piv_B \mid \alpha] = g(\sigma_A, \sigma_B) > g(\tau_A, \tau_B) = \Pr[Piv_B \mid \beta]$$

and since $\sigma_A \leq \sigma_B$, $\Pr[Piv_A \mid \alpha] \geq \Pr[Piv_B \mid \alpha]$ and so

$$\Pr[Piv_A \mid \alpha] > \Pr[Piv_B \mid \beta]$$

and by Lemma 5.4, we have $\sigma_A > \sigma_B$ which is a contradiction. ■

Lemma C.4 *If $\sigma_B < \tau_B$ and $\sigma_A < \tau_A$, then $\sigma_A > \sigma_B$.*

Proof. Suppose to the contrary that $\sigma_A \leq \sigma_B$. Lemma 5.1 implies that $\tau_A < \tau_B$.

Consider the function

$$f(x,y) = \tfrac{1}{2}e^{-x-y}\left(I_0(2\sqrt{xy}) + I_1(2\sqrt{xy})\sqrt{\frac{y}{x}}\right)$$

Notice that $\Pr[Piv_A \mid \alpha] = f(\sigma_A, \sigma_B)$ and $\Pr[Piv_A \mid \beta] = f(\tau_A, \tau_B)$. It is readily verified that

$$\begin{aligned} f_x(x,y) &= \frac{1}{x}\frac{e^{-x-y}}{2\sqrt{xy}}\left((y-x)\sqrt{xy}I_0\left(2\sqrt{xy}\right) - yI_1\left(2\sqrt{xy}\right)\right) \\ f_y(x,y) &= -\frac{e^{-x-y}}{2\sqrt{xy}}(y-x)I_1\left(2\sqrt{xy}\right) < 0 \end{aligned}$$

whenever $y > x > 0$.

For fixed x and y satisfying $y > x > 0$, and $\theta \in [0,1]$, define the function

$$h(\theta) = f(\theta x, \theta y)$$

We claim that $h'(1) < 0$. Routine calculations show that

$$h'(1) = \frac{e^{-x-y}}{2\sqrt{xy}}\left((y-x)\sqrt{xy}I_0\left(2\sqrt{xy}\right) - yI_1\left(2\sqrt{xy}\right) - y(y-x)I_1\left(2\sqrt{xy}\right)\right)$$

Now note that for all $z > 0$,

$$I_0(z) < \frac{2+z}{z}I_1(z)$$

(See Nåsell (1978)). Using this fact, we obtain

$$\begin{aligned} h'(1) &< \frac{e^{-x-y}}{2\sqrt{xy}}\left(-x + \left(\sqrt{xy} - y\right)(y-x)\right)I_1\left(2\sqrt{xy}\right) \\ &< 0 \end{aligned}$$

since $y > x$. This implies that for any x and y satisfying $y > x > 0$, and any $\theta < 1$ we have $f(\theta x, \theta y) > f(x,y)$.

This implies that for all $\theta < 1$,

$$f(\theta\tau_A, \theta\tau_B) > f(\tau_A, \tau_B) = \Pr[Piv_A \mid \beta] > \Pr[Piv_B \mid \beta]$$

Let $\theta \equiv \frac{\sigma_A}{\tau_A}$. Note that because of Lemma C.1, $\frac{\sigma_B}{\tau_B} < \frac{\sigma_A}{\tau_A} = \theta$ and so $\sigma_A = \theta\tau_A$ while $\sigma_B < \theta\tau_B$. Then the fact that $f_y < 0$ implies that

$$\Pr[Piv_A \mid \alpha] = f(\sigma_A, \sigma_B) > f(\theta\tau_A, \theta\tau_B) > \Pr[Piv_B \mid \beta]$$

Now the conclusion follows by applying Lemma 5.4.

■

Lemmas C.2, C.3 and C.4 complete the proof of Proposition 5.2.

D Proof of Proposition 5.3

Note All the results in this appendix are derived under the assumption that the distribution of voting costs is uniform.

Lemma D.1 *Suppose competence is efficient. If* $\tau_A \geq \tau_B$, *then* $(1-s)(\sigma_A - \sigma_B) > r(\tau_A - \tau_B)$.

Proof. Recall that if $\tau_A \geq \tau_B$, then $\Pr[Piv_A \mid \beta] \leq \Pr[Piv_B \mid \beta]$. Since $\lambda v < (1-\lambda)V$,

$$\frac{r\tau_A - (1-s)\sigma_A}{r\tau_B - (1-s)\sigma_B} = \frac{\lambda}{1-\lambda}\frac{v}{V}\frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_B \mid \beta]} < 1$$

■

Lemma D.2 *Suppose competence is efficient. If* $\tau_A \geq \tau_B$, *then* $\frac{\sigma_A}{\sigma_B} > \frac{\tau_A}{\tau_B}$.

Proof. Lemma D.1 implies that $\sigma_A > \sigma_B$. Also,

$$\frac{r\tau_A - (1-s)\sigma_A}{r\tau_B - (1-s)\sigma_B} < 1 < \frac{\sigma_A}{\sigma_B}$$

and from here the required inequality follows immediately. ■

Lemma D.3 *Suppose that there is a sequence of equilibria such that*

$$0 < \lim \frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} < \infty \tag{15}$$

Then for large n *along this sequence,*

$$\left(\sqrt{\sigma_A} - \sqrt{\sigma_B}\right)^2 \approx \left(\sqrt{\tau_A} - \sqrt{\tau_B}\right)^2 \tag{16}$$

Proof. When n is large, we can use (13) to write

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} \approx e^{-\left(\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2 - \left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2\right)} \frac{\frac{1}{\sqrt{4\pi\sqrt{\sigma_A\sigma_B}}}\left(1+\sqrt{\frac{\sigma_B}{\sigma_A}}\right)}{\frac{1}{\sqrt{4\pi\sqrt{\tau_A\tau_B}}}\left(1+\sqrt{\frac{\tau_B}{\tau_A}}\right)}$$

Recall that we always have $\frac{\tau_B}{\tau_A} > \frac{\sigma_B}{\sigma_A}$.

First, notice that there cannot be a sequence of equilibria along which $\lim \frac{\sigma_B}{\sigma_A} = 1$ and $\lim \frac{\tau_B}{\tau_A} > 1$. This is because in that case, $\frac{\Pr[Piv_A|\alpha]}{\Pr[Piv_A|\beta]} \to \infty$, contradicting (15). Similarly, there cannot be a sequence of equilibria along which $\lim \frac{\sigma_B}{\sigma_A} < 1$ and $\lim \frac{\tau_B}{\tau_A} = 1$. In this case, $\frac{\Pr[Piv_A|\alpha]}{\Pr[Piv_A|\beta]} \to 0$, again contradicting (15).

Next, we will argue that there cannot be a sequence of equilibria along which $\lim \frac{\sigma_B}{\sigma_A} = 1$ and $\lim \frac{\tau_B}{\tau_A} = 1$. Suppose to the contrary that both limits are 1. Since

$$\begin{aligned}
\frac{\sigma_B}{\sigma_A} &= \frac{1-\lambda}{\lambda} \frac{(1-r)\Pr[Piv_B \mid \beta] V + r \Pr[Piv_B \mid \alpha] v}{r \Pr[Piv_A \mid \alpha] V + (1-r) \Pr[Piv_A \mid \beta] v} \\
&\approx \frac{1-\lambda}{\lambda} \frac{(1-r)\Pr[Piv_A \mid \beta] \sqrt{\frac{\tau_A}{\tau_B}} V + r \Pr[Piv_A \mid \alpha] \sqrt{\frac{\sigma_A}{\sigma_B}} v}{r \Pr[Piv_A \mid \alpha] V + (1-r) \Pr[Piv_A \mid \beta] v} \\
&= \frac{1-\lambda}{\lambda} \frac{(1-r) \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} \sqrt{\frac{\tau_A}{\tau_B}} V + r \sqrt{\frac{\sigma_A}{\sigma_B}} v}{rV + (1-r) \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} v}
\end{aligned}$$

and

$$\frac{\tau_B}{\tau_A} \approx \frac{1-\lambda}{\lambda} \frac{s \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} \sqrt{\frac{\tau_A}{\tau_B}} V + (1-s) \sqrt{\frac{\sigma_A}{\sigma_B}} v}{(1-s) V + s \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} v}$$

we have that (perhaps along a further subsequence) in the limit,

$$\frac{(1-r) \lim \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} V + rv}{rV + (1-r) \lim \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} v} = \frac{s \lim \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} V + (1-s) v}{(1-s) V + s \lim \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} v}$$

Since $0 < \lim \frac{\Pr[Piv_A|\beta]}{\Pr[Piv_A|\alpha]} < \infty$, cross-multiplying leads to a contradiction (recall that $V > v$ and $r + s > 1$).

Thus along any subsequence $\lim \frac{\sigma_B}{\sigma_A} < 1$ and $\lim \frac{\tau_B}{\tau_A} \neq 1$. Thus, both $\left(\sqrt{\sigma_A} - \sqrt{\sigma_B}\right)^2$ and $\left(\sqrt{\tau_A} - \sqrt{\tau_B}\right)^2$ diverge. If their ratio is not one in the limit, then this would again contradict (15). ■

Lemma D.4 *Suppose competence is efficient and $\tau_A \geq \tau_B$. If $(1-s)\left(\sqrt{\sigma_A} - \sqrt{\sigma_B}\right)^2 < r\left(\sqrt{\tau_A} - \sqrt{\tau_B}\right)^2$, then $(1-s)(\sigma_A - \sigma_B) < r(\tau_A - \tau_B)$.*

Proof. Since $\tau_A \geq \tau_B$ implies $\sigma_A > \sigma_B$, the condition in the statement can be written as

$$\frac{\sqrt{\sigma_A} - \sqrt{\sigma_B}}{\sqrt{\tau_A} - \sqrt{\tau_B}} < \sqrt{\frac{r}{1-s}} \tag{17}$$

Now notice that if we define

$$\phi(x) = \frac{\sqrt{x} - 1}{\sqrt{x-1}}$$

then for all $x > 1$,

$$\phi'(x) = \frac{\sqrt{x} - 1}{2\sqrt{x}(x-1)^{\frac{3}{2}}} > 0$$

Since $1 \leq \frac{\tau_A}{\tau_B} < \frac{\sigma_A}{\sigma_B}$ (Lemma D.2) this implies that

$$\frac{\sqrt{\frac{\tau_A}{\tau_B}}-1}{\sqrt{\frac{\tau_A}{\tau_B}-1}} < \frac{\sqrt{\frac{\sigma_A}{\sigma_B}}-1}{\sqrt{\frac{\sigma_A}{\sigma_B}-1}}$$

or equivalently,

$$\frac{\sqrt{\sigma_A-\sigma_B}}{\sqrt{\tau_A-\tau_B}} < \frac{\sqrt{\sigma_A}-\sqrt{\sigma_B}}{\sqrt{\tau_A}-\sqrt{\tau_B}}$$

which when combined with (17) results in

$$\frac{\sqrt{\sigma_A-\sigma_B}}{\sqrt{\tau_A-\tau_B}} < \sqrt{\frac{r}{1-s}}$$

and this is the same as

$$(1-s)(\sigma_A-\sigma_B) < r(\tau_A-\tau_B)$$

■

Lemma D.5 *Suppose competence is efficient. If there is a sequence of equilibria along which*

$$0 < \lim \frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_B \mid \beta]} < \infty$$

then for large n along this sequence, $\tau_B > \tau_A$.

Proof. Suppose that there is a sequence along which $\tau_A \geq \tau_B$. We claim that $\liminf \frac{\sqrt{\sigma_A}-\sqrt{\sigma_B}}{\sqrt{\tau_A}-\sqrt{\tau_B}} > 1$. Otherwise, there is a further subsequence such that for all large n along this sequence

$$\frac{\sqrt{\sigma_A}-\sqrt{\sigma_B}}{\sqrt{\tau_A}-\sqrt{\tau_B}} < \sqrt{\frac{r}{1-s}}$$

which is the same as

$$(1-s)\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2 < r\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2$$

Lemma D.4 now implies that $(1-s)(\sigma_A-\sigma_B) < r(\tau_A-\tau_B)$, contradicting Lemma D.1.

We have thus argued that if $\tau_A \geq \tau_B$, then $\liminf \frac{\sqrt{\sigma_A}-\sqrt{\sigma_B}}{\sqrt{\tau_A}-\sqrt{\tau_B}} > 1$. This contradicts Lemma D.3. ■

Lemma D.6 *Suppose competence is efficient. If there is a sequence of equilibria along which*

$$\lim \frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_B \mid \beta]} = 0$$

then for large n along this sequence $\tau_B > \tau_A$.

Proof. Suppose to the contrary that there is a subsequence such that $\tau_B \le \tau_A$. For all large n,

$$\frac{\tau_B}{\tau_A} \approx \frac{1-\lambda}{\lambda} \frac{sV + (1-s)\frac{\Pr[Piv_B|\alpha]}{\Pr[Piv_B|\beta]} v}{(1-s)\frac{\Pr[Piv_B|\alpha]}{\Pr[Piv_B|\beta]}\sqrt{\frac{\sigma_B}{\sigma_A}}V + s\sqrt{\frac{\tau_B}{\tau_A}}v}$$

Since $\sigma_B \le \sigma_A$ and $\tau_B \le \tau_A$, along some further subsequence where both $\frac{\sigma_B}{\sigma_A}$ and $\frac{\tau_B}{\tau_A}$ converge,

$$\begin{aligned} \lim \frac{\tau_B}{\tau_A} &= \frac{1-\lambda}{\lambda} \frac{V}{v \lim \sqrt{\frac{\tau_B}{\tau_A}}} \\ \lim \left(\frac{\tau_B}{\tau_A}\right)^{\frac{3}{2}} &= \frac{1-\lambda}{\lambda} \frac{V}{v} > 1 \end{aligned}$$

which contradicts the assumption that for large n, $\tau_B \le \tau_A$. ■

Lemma D.7 *Suppose competence is efficient. If there is a sequence of equilibria along which*

$$\lim \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} = 0$$

then for large n along this sequence $\tau_B > \tau_A$.

Proof. Suppose to the contrary that $\tau_B \le \tau_A$. In this case, an argument similar to the one in the previous lemma shows that

$$\begin{aligned} \frac{\sigma_B}{\sigma_A} &\approx \frac{1-\lambda}{\lambda} \frac{(1-r)\frac{\Pr[Piv_B|\beta]}{\Pr[Piv_B|\alpha]}V + rv}{r\sqrt{\frac{\sigma_B}{\sigma_A}}V + (1-r)\frac{\Pr[Piv_B|\beta]}{\Pr[Piv_B|\alpha]}\sqrt{\frac{\tau_B}{\tau_A}}v} \\ \frac{\tau_B}{\tau_A} &\approx \frac{1-\lambda}{\lambda} \frac{r\frac{\Pr[Piv_B|\beta]}{\Pr[Piv_B|\alpha]}V + (1-r)v}{(1-r)\sqrt{\frac{\sigma_B}{\sigma_A}}V + r\frac{\Pr[Piv_B|\beta]}{\Pr[Piv_B|\alpha]}\sqrt{\frac{\tau_B}{\tau_A}}v} \end{aligned}$$

so that along a further subsequence along which $\frac{\tau_B}{\tau_A}$ and $\frac{\sigma_B}{\sigma_A}$ converge,

$$\lim \frac{\tau_B}{\tau_A} = \lim \frac{\sigma_B}{\sigma_A} = \left(\frac{1-\lambda}{\lambda}\frac{v}{V}\right)^{\frac{2}{3}} < 1$$

Thus for large n, we have that

$$\frac{r\tau_B - (1-s)\sigma_B}{r\tau_A - (1-s)\sigma_A} < \frac{\sigma_B}{\sigma_A} < 1$$

but this contradicts Lemma D.1. ■

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