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# Determinacy of Equilibria in Dynamic Models with Finitely Many Consumers*

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We consider a production economy with a finite number of heterogeneous, infinitely lived consumers. We show that, if the economy is smooth enough, equilibria are locally unique for almost all endowments. We do so by converting the infinite-dimensional fixed point problem stated in terms of prices and commodities into a finite-dimensional Negishi problem involving individual weights in a social value function. By adding artificial fixed factors to utility and production functions, we can write the equilibrium conditions equating spending and income for each consumer entirely in terms of time-zero factor endowments and derivatives of the social value function. *Journal of Economic Literature* Classification Numbers: 021, 023, 111. © 1990 Academic Press, Inc.

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## 1. Introduction

It is now well established that economic models with a finite number of goods have equilibria that are generically determinate; a specification of preferences, endowments, and technical opportunities almost always (in a sense that can be made precise) determines a finite set of equilibrium outcomes that can be observed. In models with an infinite number of goods, however, equilibria can be indeterminate. For example, Kehoe and Levine [16] provide a simple example of an overlapping generations model (without fiat money) that has a continuum of equilibria. This continuum is robust in the sense that it persists even when the basic parameters of preferences and the technology are slightly altered, and it arises in a model that is otherwise well behaved. It has no cycles and no chaos; in fact, all of the equilibria converge asymptotically to the same stable stationary state. All of the equilibria are Pareto efficient; the value of the aggregate endowment is finite; and the price sequences lie in the dual space of the commodity space. More strongly, Kehoe, Levine, Mas-Colell, and Zame [18] show that robust indeterminacy can arise when the price and commodity spaces are the same Hilbert space.

Models with indeterminacy are undesirable from a scientific point of view. Starting from the fundamentals of the economy, a theory based on this kind of model offers little guidance about what should be observed. Nor is it possible to condition on the equilibrium values that are observed and to perform comparative statics analysis because small changes in the underlying parameters can lead to large and discontinuous changes in the observed outcomes. Indeterminacy is especially troublesome in dynamic models because it undermines the interpretation of these models in terms of an equilibrium where agents trade in spot markets and form expectations about the future. If there is indeterminacy concerning the equilibrium that obtain in the present, there also is indeterminacy in the equilibria starting from future dates, and this makes the formation of expectations problematic.

In this paper, we extend the class of dynamic models that are known to have a finite number of locally unique equilibria and clarify the sense in which finiteness of the numbers of goods or consumers or both is important for this result. As noted above, models with a finite number of goods are generically determinate, but this is of no help for the study of infinite-horizon dynamic models. Moreover, we know from examples that a finite number of goods is not necessary for determinacy. A model with a single representative agent is determinate regardless of the number of goods in the model; the equilibrium quantities are solutions to a concave maximization problem, and, under mild strict concavity assumptions, they are unique. The Kehoe–Levine [16] overlapping generations example shows that indeterminacy can arise in a model with both an infinite number of goods and an infinite number of individuals. The results presented here suggest that this kind of double infinity of goods and consumers is the crucial element in this example. In particular, we show that what matters for determinacy in the usual representative agent dynamic model is not that there be a single agent in the economy. Rather, it is sufficient for the number of agents to be finite.

Proofs of determinacy in a model with a finite number of goods proceed by examining the properties of a finite number of supply-equals-demand equations is the same as the number of undetermined prices, there is almost always a finite number of solutions to these equations. Counting equations and unknowns does not work, however, when there is an almost always a finite number of solutions to these equations. Counting equations and unknowns does not work, however, when there are an infinite number of equations and variables. What we exploit is a duality relationship between goods and individuals. We reduce the specification of the equilibrium to a finite number of equations, one for each individual, that equate the value of consumption with the value of the individual's endowment. These equations depend on a set of welfare weights that are the analogs of prices. The weights are assigned to individuals in a Pareto optimization problem that determines consumption for each individual by maximizing the weighted sum of individual utilities. Our analysis of determinacy then proceeds as in the usual finite-dimensional case.

Our results extend those of Muller and Woodford [23], who consider production economies with both finitely and infinitely lived consumers. They show that there can be no indeterminacy if the influence of the infinitely lived consumers is sufficiently large. Their results are local, however, and concern only equilibria that converge to a particular stationary state. We prove a global theorem: for a given starting capital stock, there are only finitely many equilibria. (We do not, however, permit an infinite number of finitely lived consumers, as do Muller and Woodford.)

We assume that markets are complete and that technology and preferences are convex. Consequently, the behavior of equilibria in our model can be characterized by the properties of a value function. This is because the second theorem of welfare economics holds: any Pareto efficient allocation can be decentralized as a competitive equilibrium with transfer payments. If the preferences of consumers can be represented by concave utility functions, then an equilibrium with transfers can be calculated by maximizing a weighted sum of the individual utility functions, subject to the feasibility constraints implied by the aggregate technology and the initial endowments. Showing that an equilibrium exists is equivalent to showing that there exists a vector of welfare weights such that the transfer payments needed to decentralize the resulting Pareto efficient allocation are zero. This approach has been pioneered by Negishi [24] and applied to dynamic models by Bewley [3, 4]. Using this approach, Kehoe and Levine [14] have considered the regularity properties of an infinite-horizon economy without production.

In general, calculating the transfer associated with a given set of weights requires the complete calculations of equilibrium quantities and prices. In a dynamic model with an infinite number of commodities, this can be awkward. To simplify the calculation, we adopt an alternative strategy based on the simple geometric observation that any convex set in $\mathbf{R}^n$ can be interpreted as the cross section of a cone in $\mathbf{R}^{n+1}$. To exploit this fact, we add a set of artifical fixed factors to the economy and include them as arguments of the weighted social value function. These factors are chosen so that the augmented utility and production functions are homogeneous of degree one. Thus, the usual problem of choosing a point on the frontier of a convex utility possibility set is converted into a problem of choosing a point from a cone of feasible values for utility. This extension has theoretical advantages analogous to those that arise when a strictly concave production function is converted into a homogeneous-of-degree-one function by the addition of a fixed factor. When the technicology for the firm is a cone, profits and revenues are completely accounted for by factor payments. Analogously, making the social value function homogeneous of degree one simplifies the accounting necessary to keep track of the transfers associated with any given Pareto efficient allocation. The present value of income and expenditure for each individual can be calculated directly from an augmented list of endowments and from the derivatives of the augmented social value function without explicitly calculating the dynamic paths for prices or quantities. This is the framework for studying multiagent intertemporal equilibria developed by Kehoe and Levine [15].

In such a setting, equilibria are equivalent to the zeros of a simple, finite-dimensional system of equations involving the derivatives of the social value function and the endowments. Intuition says that, since both the number of equations and the number of unknowns in this system are equal to the number of agents, equilibria ought to be determinate. To do the usual kind of regularity analysis, however, the functions involved in the system of equations that determines the equilibria must be continuously differentiable. Because these functions are combinations of derivatives of the social value function, they are continuously differentiable if the value function is twice continuously differentiable. Thus, we show how to reduce an equilibrium problem to a problem in growth theory: How smooth is the value function? Roughly, the result is the same as in the finite case. If the economy (in this case, the value function) is smooth enough, equilibria are generically determinate. Unlike the finite-horizon case, however, we have no simple and general assumptions on preferences and technology that guarantee the economy is smooth enough.

In our basic model—a one-sector, neoclassical growth model—smoothness of the value functions may be deduced from a global turnpike theorem. With multiple capital stocks, twice-continuous differentiability of the value function is not known to follow from smoothness of utility and production possibilities, except in special cases. If the discount factor is sufficiently close to one, then, as Araujo and Scheinkman [1] show, the value function must be twice continuously differentiable; if the disjoint factor is sufficiently close to zero, then, as Boldrin and Montrucchio [8] show, the value function must be twice continuously differentiable. In the latter case, a global turnpike theorem is not true, and Boldrin and Montrucchio [7] and Deneckere and Pelikan [11] have shown that both cycles and chaos can occur with small discount factors. In fact, Boldrin and Montrucchio [7] describe a general method for constructing examples with $C^\infty$ smoothness and chaos. Consequently, determinacy does not rest on a turnpike theorem but on the rather different assumption of a twice continuously differentiable value function.

In the case of intermediate values of the discount factor, relatively little is known. Boldrin and Montrucchio [8] provide some conditions sufficient for a $C^2$ value function. In addition, Kehoe, Levine, and Romer [17] show that, regardless of the discount factor, there are at most finitely many equilibria that converge to a nondegenerate steady state. Moreover, our general methods apply to stochastic as well as deterministic complete contingent claims economies (see Kehoe and Levine [15]), and results due to Blume, Easley, and O'Hara [5] imply that a small amount of the right kind of uncertainty leads to smooth value functions. These results can provide an alternative direction for proving determinacy results.

In Section 2 we set up a simple one-sector, multiperson economy. Section 3 characterizes equilibria as social optima without transfer payments. Section 4 analyzes properties of the savings function of the economy. Section 5 shows how differentiability of the savings function implies determinacy of equilibria. Finally, Section 6 discusses extensions to multisector models.

## 2. The Basic Model

Consider a simple $m$-person neoclassical growth model. The preferences of each consumer take the usual additively separable form, discounted by the common factor $\beta$, where $0<\beta<1$. Consumption of the single perishable good by consumer $i$ at time $t$ is denoted by $c_{it}$. The utility function for consumer $i$, $i=1, ..., m$, is then $\sum_{t=0}^{\infty} \beta^t u_i(c_{it})$. The initial endowment of capital, the single reproducible productive factor, is $\bar{k}_0>0$, and

$\theta_i > 0$ is the share owned by consumer $i$. Obviously, $\sum_{i=1}^{m} \theta_i = 1$, and $\theta_i \bar{k}_0$ is the endowment of consumer $i$. The economy also has $\bar{x} > 0$ units of labor, a perishable productive factor in each period; $\phi_i > 0$ is the share of total labor owned by consumer $i$. Again, $\sum_{i=1}^{m} \phi_i = 1$, and $\phi_i \bar{x}$ is the endowment of consumer $i$. If $q_t$ denotes the output of consumption and $k_t$ and $x_t$ denote the inputs of capital and labor, the technology is described by an aggregate production function $q_t + k_{t+1} = G(k_t, x_t)$.

Notice that we have implicity assumed that all consumers discount utility using the same factor $\beta$. There is no difficulty in proving the existence of equilibrium, or even in establishing a global turnpike theorem, if we allow heterogeneous discount factors $\beta_i$ (see Coles [10]). In this case all consumers with discount factors less than the maximum discount factor consume nothing in the steady state. We could then need to put additional assumptions on the utility function to ensure that, as consumption falls, the value function remains twice continuously differentiable. Such an approach is possible (see Kehoe and Levine [14] and Muller and Woodford [23]), but technically tedious. We follow the alternative of assuming homogeneous discount factors here.

We next specify the properties assumed for the preferences and technology. The assumptions concerning continuity, monotonicity, and concavity are standard.

*Assumption* 1. For $1, ..., m$, $u_i: \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$ is concave, strictly increasing, and continuous. On the strictly positive orthant, $\mathbf{R}_{++}$, $u_i$ is smooth (infinitely many times continuously differentiable) and $\partial u_i/\partial c > 0$, $\partial^2 u_i/\partial^2 c < 0$. Moreover, $\lim_{c \to 0} \partial u_i(c)/\partial c = +\infty$.

In the statement that $u_i$ is continuous, we are using the natural topology on $\mathbf{R} \cup \{-\infty\}$, that is, the one generated by adding open intervals of the form $[-\infty, a)$ to the usual topology on $\mathbf{R}$. For example, the functions $u(c) = \ln c$ and $u(c) = (c^\rho - 1)/\rho$, for $\rho \leqslant 1$, both satisfy the conditions of this assumption.

*Assumption* 2. The production function $G: \mathbf{R}_+^2 \to \mathbf{R}_+$ is homogeneous of degree one, concave, and continuous. On the strictly positive orthant, $\mathbf{R}_{++}^2$, $G$ is smooth, and $\partial G/\partial k > 0$, $\partial G/\partial x > 0$, $\partial^2 G/\partial k^2 < 0$. Moreover, $G(0, x) = 0$, $\lim_{k \to 0} \partial G(k, \bar{x})/\partial k = +\infty$, and $\limsup_{k \to \infty} G(k, \bar{x})/k < 1$.

In what follows, smoothness plays an important role. To ensure that it holds, we need to know not only that the relevant functions are smooth but also that production and consumption plans are strictly interior. This follows from the infinite steepness conditions, $\partial u_i(0)/\partial c = +\infty$ and $\partial G(0, \bar{x})/\partial k = \infty$. Finally, by assuming $\limsup_{k \to \infty} G(k, \bar{x})/k < 1$, we ensure that the capital stock must remain bounded.

## 3. Characterization of Equilibrium

A competitive equilibrium for this model consists of a sequence $p_0, p_1, \ldots$ of prices for the consumption good; a price $r$ for the initial capital stock; a sequence $w_0, w_1, \ldots$ of prices for labor; a consumption allocation $c_{i0}, c_{i1}, \ldots$ for each consumer $i$; a sequence of capital stocks $k_0, k_1, \ldots$; a sequence of labor inputs $x_0, x_1, \ldots$; and a sequence of outputs of the consumption good $q_0, q_1, \ldots$. Given the prices $p_t$, $w_t$, and $r$, the consumption allocation $c_{it}$ must solve the utility maximization problem for consumer $i$,

$$\max \sum_{t=0}^{\infty} \beta^t u_i(c_{it}),$$

subject to

$$\sum_{t=0}^{\infty} p_t c_{it} \leqslant \theta_i r \bar{k}_0 + \phi_i \sum_{t=0}^{\infty} w_t \bar{x}.$$

Furthermore, given the prices $p_t$, the production plans $k_t$, $x_t$, and $q_t$ must maximize profits,

$$\max \sum_{t=0}^{\infty} (p_t q_t - w_t x_t) - r k_0,$$

subject to

$$q_t + k_{t+1} \leqslant G(k_t, x_t). \qquad t = 0, 1, \ldots.$$

Finallt, demand must equal supply for the consumption good and labor in every period and for the initial capital stock:

$$\sum_{i=1}^{m} c_{it} = q_t, \qquad t = 0, 1, \ldots$$

$$x_t = \bar{x}$$

$$k_0 = \bar{k}_0.$$

As is usual in such capital theory problems, the construction of a competitive equilibrium is accomplished by the solution of a social optimization problem. The necessary conditions for this problem guarantee the existence of a set of shadow prices that satisfy certain properties. It is then an easy matter to show that these prices, along with the optimal quantities, satisfy the sufficient conditions for the optimization problems of the consumer and the firm.

Consider the social planning problem of determining a Pareto efficient consumption allocation and production sequence. Given nonnegative welfare weights $(\alpha_1, \alpha_2, ..., \alpha_m)$, we maximize a weighted sum of the individual consumers' utilities subject to feasibility constraints,

$$\max \sum_{i=1}^{m} \alpha_i \sum_{t=0}^{\infty} \beta^t u_i(c_{it}),$$

subject to

$$\sum_{i=1}^{m} c_{it} + k_{t+1} \leqslant G(k_t, x_t), \qquad t = 0, 1, ...$$

$$x_t \leqslant \bar{x}, \qquad t = 0, 1, ...$$

$$k_0 \leqslant \bar{k}_0$$

$$k_t \geqslant 0, \qquad t = 0, 1, ... .$$

Using results that are analogous to the Kuhn–Tucker Theorem for finite maximization problems, we can express the necessary conditions for this problem as a set of intertemporal optimization conditions and a transversality condition at infinity. Let $p_t$ be the Lagrange multiplier applied to the constraint on output in each period, let $w_t$ be the multiplier on the constraint on $x_t$, and let $r$ be the multiplier on the initial stock of capital. The Lagrangian for this problem is

$$\mathscr{L} = \sum_{i=1}^{m} \alpha_i \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) + r(\bar{k}_0 - k_0)$$
$$+ \sum_{t=0}^{\infty} \left[ w_t(\bar{x} - x_t) + p_t \left( G(k_t, x_t) - k_{t+1} - \sum_{i=1}^{m} c_{it} \right) \right].$$

Since the optimal qualities $c_{it}$, $k_t$, and $x_t$ are strictly positive, the intertemporal optimization conditions follow by setting derivatives of $\mathscr{L}$ equal to zero:

$$\alpha_i \beta^t \, \partial u_i(c_{it})/\partial c - p_t = 0, \qquad i = 1, ..., m; \quad t = 0, 1, ...$$

$$p_0 \, \partial G(k_0, x_0)/\partial k - r = 0,$$

$$-p_{t-1} + p_t \, \partial G(k_t, x_t)/\partial k = 0, \qquad t = 1, 2, ...$$

$$p_t \, \partial G(k_t, x_t)/\partial x - w_t = 0, \qquad t = 0, 1, ...$$

$$x_t = \bar{x}, \qquad t = 0, 1, ...$$

$$k_0 = \bar{k}_0.$$

The transversality condition at infinity is

$$\lim_{t \to \infty} p_t k_{t+1} = 0.$$

See Weitzman [29] and Romer and Shinotsuka [25] for discussions of the role of this condition.

The sufficient conditions for the problem of the consumer and the problem of the firm can be derived analogously. Let $\lambda_i$ denote the multiplier on the budget constraint for consumer $i$ when faced with prices $p_t$. The sufficient conditions for an optimal consumption sequence are

$$\beta^t \, \partial u_i(c_{it})/\partial c - \lambda_i p_t = 0, \qquad t = 0, 1, \ldots$$

combined with the requirement of overall budget balance,

$$\sum_{t=0}^{\infty} p_t c_{it} = \theta_i r \bar{k}_0 + \phi_i \sum_{t=0}^{\infty} w_t \bar{x}.$$

Note that the left-hand side of this expression—expenditure on consumption—is strictly decreasing in $\lambda_i$ and the right-hand side is given, so this expression can be thought of as determining $\lambda_i$.

Let $\mu_t$ denote the Lagrange multiplier associated with the constraint faced by the firm. Sufficient conditions for the firm's maximization problem are once again a set of intertemporal conditions and a transversality condition at infinity. Again using the fact that the optimal quantity choices are interior, we can derive the intertemporal conditions by setting derivatives of this expression equal to zero:

$$\begin{aligned}
p_t - \mu_t &= 0, && t = 0, 1, \ldots \\
-r + \mu_0 \, \partial G(k_0, x_0)/\partial k &= 0 && \\
-\mu_{t-1} + \mu_t \, \partial G(k_t, x_t)/\partial k &= 0, && t = 1, 2, \ldots \\
-w_t + \mu_t \, \partial G(k_t, x_t)/\partial x &= 0, && t = 0, 1, \ldots .
\end{aligned}$$

The transversality condition is

$$\lim_{t \to \infty} \mu_t k_{t+1} = 0.$$

These conditions can be simplified to

$$\begin{aligned}
&\partial G(k_0, \bar{x})/\partial k_0 = r/p_0 && \\
&\partial G(k_t, \bar{x})/\partial k_t = p_{t-1}/p_t, && t = 1, 2, \ldots \\
&\partial G(k_0, \bar{x})/\partial x = w_t/p_t, && t = 0, 1, \ldots \\
&\lim_{t \to \infty} p_t k_{t+1} = 0. &&
\end{aligned}$$

Comparing the necessary conditions for the weighted social optimization problem with the sufficient conditions for the consumer and firm problems, we observe that the quantities from a competitive equilibrium are Pareto efficient. They solve the social optimization problem when the weights $\alpha_i$ are chosen to satisfy $\alpha_i = 1/\lambda_i$. This is simply the first welfare theorem for this economy. Note, too, that for any arbitrary weights $\alpha_i$, the quantities from the social optimization problem can be decentralized as a competitive equilibrium with transfers. All that is required is to adjust the multiplier $\lambda_i$ representing the marginal utility of income for each consumer so its reciprocal equals that consumer's weight $\alpha_i$. As we have remarked above, $\lambda_i$ for each consumer varies monotonically with the income allocated to consumer $i$. This procedure is, of course, an application of the second welfare theorem for this economy. The appropriate transfer to each consumer is the amount that just allows the consumer to afford the consumption stream allocated by the social optimization problem. Thus, for given weights $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$, the required transfers are

$$\sum_{t=0}^{\infty} p_t(\alpha)\, c_{it}(\alpha) - \theta_i r(\alpha)\, \bar{k}_0 - \phi_i \sum_{t=0}^{\infty} w_t(\alpha) \bar{x}, \qquad i = 1, ..., m.$$

For this economy, a competitive equilibrium in the usual sense corresponds to a set of weights $\alpha$ such that these transfer to zero.

In principle, we have all that we need to consider the regularity properties of this economy: by equating the transfer payments for each of the $m$ consumers to 0, we have $m$ equations in the $m$ unknowns, $(\alpha_1, \alpha_2, ..., \alpha_m)$. In practice, this is not a useful system of equations to work with because the equations require the calculation of the infinite set of quantities $c_{it}(\alpha)$ and an infinite list of prices $p_t(\alpha)$, $w_t(\alpha)$ for each choice of the vector $\alpha$. One could attempt to explicitly characterize the dependence of these infinite-dimensional vectors on $\alpha$. In the next section we show how this step can be avoided by the use of a suitably chosen value function. Of course, the infinite-dimensional nature of the problem does not disappear. Rather, it is embodied in the properties of the value function, which is the result of solving the social planning problem—an easier infinite-dimensional problem than the original equilibrium problem.

## 4. The Savings Function

Let us now characterize solutions to the social planning problem, and competitive equilibria, in dynamic programming terms. Given an aggregate endowment of capital $k_0$, a labor supply $x$, and a vector of nonnegative welfare weights $\alpha$, we define a value function $V(k_0, x, \alpha)$ as the maximum of

$$\sum_{i=1}^{m} \alpha_i \sum_{t=0}^{\infty} \beta^t u_i(c_{it})$$

subject to the constraints

$$\sum_{i=1}^{m} c_{it} + k_{t+1} \leqslant G(k_t, x), \qquad t = 1, 2, \ldots .$$

The envelope theorem allows us to treat the derivative $\partial V(k_0, x, \alpha)/\partial k_0$ as the price for capital $r$ and then use it to calculate the value of the capital endowment $\theta_i \bar{k}_0$ for each individual. Similarly, we can show that

$$\partial V(k_0, \bar{x}, \alpha)/\partial x = \sum_{t=0}^{\infty} w_t,$$

so we can calculate the present value of the labor endowment $\phi_i \bar{x}$ for each consumer. To calculate the transfers associated with these weights, we must also calculate each consumer's expenditure.

To calculate individual expenditures, we must introduce an accounting device. We first show that strictly concave utility functions can be made homogeneous of degree one. In production theory, a decreasing-returns technology can be converted into a constant-returns technology by introducing a fixed factor to act as an accounting device to track producer surplus—the difference between revenue and expenditure. (See, for example, McKenzie [19].) A similar factor can be used to account for consumer surplus—the difference between utility and expenditure. We introduce an additional, person-specific, fixed utility factor $y_i$ for each agent and endow agent $i$ with the entire aggregate supply of one unit of factor $i$. (For simplicity of notation, we make no distinction between the individual's holdings of factor $y_i$ and the aggregate endowment.) As in production theory, for $y_i > 0$, we define an augmented utility function $U_i(c, y_i) = y_i u_i(c/y_i)$. We now define a value function $V(k_0, x, y, \alpha)$ as the maximum of the weighted sum of the augmented utility functions subject to the augmented technology.

If we let $c_{it}$ denote the optimal consumption of agent $i$ at time $t$, the first-order conditions from the maximization problems imply the equality

$$\beta^t \alpha_i \, \partial U_i(c_{it}, y_i)/\partial c = \beta^t \alpha_j \, \partial U_j(c_{jt}, y_j)/\partial c.$$

As a result, weighted discounted marginal utility for any consumer can be used as a present-value price for consumption at time $t$. The only difference from the usual representative consumer framework is that the weights $\alpha$ convert the individual marginal utility prices into a social marginal value price. We can then evaluate the expenditure of consumer $i$ in period $t$ as $c_{it}$ multiplied by this price. Using the properties of homogeneous functions, we can decompose period $t$ utility for consumer $i$ into the sum of a term of this form and an analogous term involving the added utility factor:

$$U_i(c_{it}, y_i) = c_{it}\,\partial U_i(c_{it}, y_i)/\partial c + y_i\,\partial U_i(c_{it}, y_{it})/\partial y.$$

If the term involving the utility factor is interpreted as a measure of consumer surplus, expenditure on goods in period $t$ is simply utility minus consumer surplus. Using the envelope theorem, we can then calculate the present value of consumer surplus for agent $i$ as the derivative of the social value function $V(k_0, x, y, \alpha)$ with respect to $y_i$, multiplied by the endowment $y_i$:

$$y_i\,\partial V(k_0, x, y, \alpha)/\partial y_i = \sum_{t=0}^{\infty} \beta^t \alpha_i y_i\,\partial U_i(c_{it}, y_i)/\partial y_i.$$

Similarly, we can calculate the discounted sum of utility for consumer $i$, measured in social value units, as the derivative (we show below that $V$ is differentiable) of the social function with respect to $\alpha_i$, multiplied by $\alpha_i$:

$$\alpha_i\,\partial V(k_0, x, y, \alpha)/\partial \alpha_i = \sum_{t=0}^{\infty} \beta^t \alpha_i U_i(c_{it}, y_i).$$

Then the present value of expenditure by agent $i$ is simply the difference

$$\alpha_i\,\partial V(k_0, x, y, \alpha)/\partial \alpha_i - y_i\,\partial V(k_0, x, y, \alpha)/\partial y_i.$$

The transfer to agent $i$ necessary to support this equilibrium is zero if and only if this expenditure is equal to the time-zero value of the agent's endowment,

$$\theta_i k_0\,\partial V(k_0, x, y, \alpha)/\partial k_0 + \phi_i \bar{x}\,\partial V(k_0, x, y, \alpha)/\partial x.$$

Formally, equality of these two expressions can be interpreted in terms of an augmented economy where trade in the utility factors $y_i$ actually takes place. In this case, this equality can be interpreted as a requirement that the value of the augmented endowment for agent $i$, $\theta_i \bar{k}_0\,\partial V/\partial k_0 + \phi_i \bar{x}\,\partial V/\partial x + y_i\,\partial V/\partial y_i$, equal the amount of social utility purchased, $\alpha_i\,\partial V/\partial \alpha_i = \alpha_i \sum_{t=0}^{\infty} \beta^t U_i$.

It is useful to define a net savings function $s_i$ for consumer $i$ as

$$\begin{aligned} s_i(k_0, \theta, \phi, \alpha) = {} & \theta_i k_0\,\partial V(k_0, \bar{x}, 1, \alpha)/\partial k_0 + \phi_i \bar{x}\,\partial V(k_0, \bar{x}, 1, \alpha)/\partial x \\ & + y_i\,\partial V(k_0, \bar{x}, 1, \alpha)/\partial y_i - \alpha_i\,\partial V(k_0, \bar{x}, 1, \alpha)/\partial \alpha_i. \qquad (1) \end{aligned}$$

For a given set of welfare weights $\alpha$, the transfer for each consumer needed to support the social optimum as a competitive equilibrium is the negative of the net savings for that consumer. A competitive equilibrium is therefore equivalent to a vector of weights $\alpha$ such that the vector $s(\bar{k}_0, \theta, \phi, \alpha) = 0$.

To calculate equilibria, we need to analyze the savings function. Our goal is to prove the following proposition.

PROPOSITION 1. *Under Assumptions* 1 *and* 2,

(a) *The net savings function* $s(k_0, \theta, \phi, \alpha)$ *is homogeneous of degree one in* $\alpha$.

(b) $\sum_{i=1}^{m} s_i(k_0, \theta, \phi, \alpha) = 0$.

(c) *For each* $k_0$, $\theta$, *and* $\phi$, $\lim_{\alpha_i \to 0} s_i(k_0, \theta, \alpha) > 0$.

(d) *The savings function s is continuously differentiable*; *it is affine in both* $\theta$ *and* $\phi$, *and* $D_\theta s$ *is diagonal and nonsingular.*

An implication of this proposition is that the functions $s_i(\alpha)/\alpha_i$ satisfy the same formal properties as the excess demand functions of a static pure exchange economy with $m$ goods. This observation leads immediately to the conclusion that an equilibrium exists.

In proving Proposition 1, note that in part (d), the dependence of $s$ on $\theta$ is obvious from the definition of the savings function (1): $s$ is affine in $\theta$ and $D_\theta s$ is a diagonal matrix with diagonal entries $k_0\, \partial V(k_0, \bar{x}, 1, \alpha)/\partial k_0$. Moreover, since utility and production are strictly monotone and endowments are strictly positive, these entries are strictly positive. Similarly, $s$ is affine in $\phi$.

The proof of the remaining parts of Proposition 1 follows by using dynamic programming to characterize the value function. To solve the social optimization problem, we first solve the problem in period $t$ for a given $k_t$ and $k_{t+1}$. Let the vector of weights $\alpha$ be fixed. Define $w(C, y, \alpha)$ as the maximum of

$$\sum_{i=1}^{m} \alpha_i U_i(c_i, y_i)$$

subject to

$$\sum_{i=1}^{m} c_i = C.$$

The following result follows immediately from Assumption 1:

LEMMA 1. *The function w is concave in* $(C, y)$, *convex in* $\alpha$, *strictly increasing in C and* $\alpha$, *and continuous. On the strictly positive orthant, w is* *smooth and* $\partial w/\partial C > 0$, $\partial^2 w/\partial C^2 < 0$. *Moreover,* $\partial w(0, y, \alpha)/\partial C = +\infty$. *The function* $w$ *is homogeneous of degree one in* $(C, y)$ *and separately homogeneous of degree one in* $\alpha$.

Using the function $w$, we can write social value in period $t$ as

$$v(k_t, k_{t+1}, x, y, \alpha) = w(G(k_t, x) - k_{t+1}, y, \alpha).$$

The social present-value function $V$ then satisfies the dynamic programming relationship

$$V(k_t, x, y, \alpha) = \max_{k_{t+1}} v(k_t, k_{t+1}, x, y, \alpha) + \beta V(k_{t+1}, x, y, \alpha).$$

*Proof of Proposition* 1. Note that $v$ is concave and homogeneous of degree one in $(k_t, k_{t+1}, x, y)$ and convex and homogeneous of degree one in $\alpha$. It follows directly that $V$ shares these same properties. An argument of Benveniste and Scheinkman [2] implies that it is also continuously differentiable. Examining (1), we see that $s_i$ is made up of terms where a constant is multiplied by $\partial V/\partial z$, for arguments $z$ other than $\alpha$, and of a term $\alpha_i\, \partial V/\partial \alpha_i$. In either case, each term is homogeneous of degree one, since $V$ is homogeneous of degree one in $\alpha$. This proves part (a) of the proposition. Moreover, since $V$ is continuously differentiable, homogeneity of $V$ of degree one in $k_t$, $x$, and $y$, and $\sum_i \theta_i = 1$, $\sum_i \phi_i = 1$ imply that the terms

$$\theta_i k_0\, \partial V/\partial k_0 + \phi_i \bar{x}\, \partial V/\partial x + y_i\, \partial V/\partial y_i$$

add up to $V(k_t, x, y, \alpha)$. Homogeneity of $V$ of degree one in $\alpha$ implies that the terms $\alpha_i\, \partial V/\partial \alpha_i$ add up to $V$ as well. This proves part (b).

Next, let $\alpha^l$ be a sequence in the interior of the positive orthant converging to a point $\alpha$ such that $\alpha_i = 0$ and $\alpha_j \neq 0$. If $c_i^l$ and $c_j^l$ denote the corresponding optimal consumption choices in the definition of $w$, infinite steepness on the boundary of the utility functions implies that $c_i^l$ and $c_j^l$ are strictly positive and that the equality

$$\alpha_i^l\, \partial u_i(c_i^l)/\partial c_i = \alpha_j^l\, \partial u_j(c_j^l)/\partial c_j \tag{2}$$

holds for all $l$. By an application of the maximum theorem (see, for example, Hildenbrand [13]), $s$ is a continuous function, since the mapping that sends $\alpha$ to the vector of optimal consumptions is single valued. From the definition of $w$, it is clear that the optimal value for $c_i$ is 0, since $\alpha_i = 0$. By continuity, $c_i^l$ converges to zero while $\alpha_j^l > 0$ clearly implies $c_j^l \neq 0$. Using the equality (2) noted above, this implies that

$$\lim_{l \to \infty} \alpha_i^l c_i^l\, \partial u_i(c_i^l)/\partial c_i = 0.$$

That is, the expenditure on consumption goods allowed consumer $i$ goes to 0 as the consumer's weight in social utility goes to 0.

By the envelope theorem, we know that the last term in the definition of $s$ is simply the product of the utility weights times the present discounted utility for each consumer. Using the homogeneity of the augmented utility functions $U_i(c, y_i)$, we can combine the last two terms in $s$ and express $s_i$ as

$$s_i(k_0, \theta, \phi, \alpha) = \theta_i k_0\, \partial V(k_0. \bar{x}, 1, \alpha)/\partial k_0 + \phi_i \bar{x}\, \partial V(k_0, \bar{x}, 1, \alpha)/\partial x$$
$$- \alpha_i \sum_{t=0}^{\infty} \beta^t c_{it}\, \partial u_i(c_{it})/\partial c_{it}.$$

By the above argument, the last term in this expression goes to 0 as $\alpha_i^l$ goes to 0. From the definition of $w$ and the envelope theorem, the first and second derivatives of $V$ can be expressed in terms of the marginal utility of agent $j$ and hence are continuous as $l \to \infty$. Therefore, $s$ approaches

$$\theta_i k_0\, \partial V(k_0, \bar{x}, 1, \alpha)/\partial k_0 + \phi_i\, \partial V(k_0, \bar{x}, 1, w)/\partial x.$$

Note that this last term is simply the value of consumer $i$'s endowment of capital and labor. By assumption, $\theta_i$, $\phi_i > 0$. Since $G$ and $u_j$ were assumed to be strictly increasing, every component of $\partial V/\partial k_0$ is strictly positive and $s_i$ is greater than 0.

It remains to show that $s$ is continuously differentiable. From (1), it clearly suffices to show that $V$ is twice continuously differentiable. In this one-dimensional example, $C^2$ differentiability of $v$ follows from basic properties of the social optimization problem. Given $\theta$, $\phi$, and $\alpha$, the maximization problem is a standard one-sector growth problem with a period objective function $w$. By Lemma 1, this satisfies standard properties. It is well known that such a problem satisfies a global turnpike property and has no unit roots. See, for example, Harris [12, pp. 34–45] or Cass [9]. In addition, $v$ is strictly concave in $(k_t, k_{t+1})$ and

$$\frac{\partial^2 v}{\partial k_t\, \partial k_{t+1}}(k_t, k_{t+1}, x, y, \alpha)$$
$$= -\frac{\partial^2 w}{\partial c^2}(G(k_t, x) - k_{t+1}, y, \alpha)\frac{\partial G}{\partial k_t}(k_t, x) > 0.$$

Note that $\partial^2 v/\partial k_t\, \partial k_{t+1}$ could be negative if there were joint production, that is, if the consumption good and the investment good were complements rather than perfect substitutes in production. The implications of this possibility are discussed in Section 6. Araujo and Scheinkman [1] show that, in a situation where it is positive and there is a global turnpike with no unit roots, $V$ is twice continuously differentiable with respect to $k_t$ and $\beta$. A straightforward extension of their argument shows that it is also jointly twice continuously differentiable with respect to $k_t$, $x$, $y$, and $\alpha$. Let $k$ denote the infinite vector $(k_1, k_2, ...)$ in $l_\infty$, the space of bounded sequences. The optimal path can be characterized by first-order conditions $\xi(k, k_0, x, y, \alpha) = 0$, where $\xi(\cdot, k_0, x, y, \alpha)\colon l_\infty \to l_\infty$. Under the stated conditions, Araujo and Scheinkman [1] show that $\xi$ is continuously differentiable in $k$ and $k_0$. Kehoe, Levine, and Romer [17, Appendix] show that this extends in a straightforward way to cover other parameters such as $x$, $y$, and $\alpha$. Araujo and Scheinkman also show that, under the stated conditions, the derivative of $\xi$ with respect to $k$ is nonsingular; it follows from the implicit function theorem that the optimal $k$ is a continuously differentiable function of $k_0$, $x$, $y$, and $\alpha$. Finally, observe that, by the envelope theorem, the first derivatives of $V$ are continuously differentiable functions of $k$, $k_0$, $x$, $y$, and $\alpha$. The twice-continuous differentiability of $V$ then follows from the fact that the composition of continuously differentiable functions is continuously differentiable. ∎

## 5. Determinacy of Equilibria

By considering savings functions $s$, we have reduced the problem of finding an equilibrium to the problem of finding an $m$-vector $\alpha$ that solves the $m$ equations $s(k_0, \theta, \phi, \alpha) = 0$. In this section we use that fact to prove that for almost all endowments, there is a finite (and odd) number of equilibria.

From property (b) of Proposition 1, we see that $\sum_{i=1}^m s_i = 0$. Consequently, it suffices to solve the system $s^{-m} = (s_1, s_2, ..., s_{m-1})$, with one equation deleted. Moreover, by property (a), we may restrict $\alpha$ to lie on the unit simplex. Since $\sum_{i=1}^m \theta_i = 1$, we may simply set $\theta_m = 1 - \sum_{i=1}^{m-1} \theta_i$ and let $\theta^{-m} = (\theta_1, ..., \theta_{m-1})$. (Recall that $\theta_i$ is consumer $i$'s share of the total stock of capital.) From property (d), we see that for fixed $k_0$ and $\phi$ we may solve $s^{-m}(k_0, \theta, \phi, \alpha) = 0$ to find

$$\theta^{-m} = f(\alpha),$$

where $f$ is continuously differentiable. Indeed,

$$\begin{aligned} f_i(\alpha) = &-(\phi_i \bar{x}\, \partial V(\bar{k}_0, \bar{x}, 1, \alpha)/\partial x + y_i\, \partial V(\bar{k}_0, \bar{x}, 1, \alpha)/\partial y_i \\ &- \alpha_i\, \partial V(\bar{k}_0, \bar{x}, 1, \alpha)/\partial \alpha_i) \\ &\div (\bar{k}_0\, \partial V(\bar{k}_0, \bar{x}, 1, \alpha)/\partial k_0). \end{aligned}$$

Note that, for some values of $\alpha$, some components of $\theta^{-m}$ may be negative. Indeed, from property (c), we see that if $\alpha_i$ is zero, $\theta_i$ is always negative. Of course, no such $\alpha$ can arise in equilibrium.

For a two-person economy ($m=2$), we sketch $f$ in Fig. 1. In this case, the existence of an eqauilibrium for each $0 \leqslant \theta_1 \leqslant 1$ follows from $f<0$ when $\alpha_1=0$ and $f>1$ when $\alpha_1=1$. In the general case, existence follows from Brouwer's fixed-point theorem as, for example, in Varian [28].

Let us call an equilibrium $\alpha$ *regular* if the $(m-1)\times(m-1)$-dimensional derivative matrix of $f$ with respect to $\alpha$, $D_\alpha f(\alpha)$ is nonsingular. We call a value $\theta^{-m}$ *regular* if every corresponding equilibrium $\alpha$, characterized by $\theta^{-m}=f(\alpha)$, is regular.

For fixed $\theta^{-m}$, can there be a sequence of equilibria $\alpha^n \neq \alpha$ converging to a regular equilibrium $\alpha$? To see that there cannot be, suppose to the contrary that there is such a sequence. Clearly, $(\alpha^n-\alpha)/\|(\alpha^n-\alpha)\|$ has a convergent subsequence converging to a vector $d$ with unit length. Moreover, $f(\alpha^n)=f(\alpha)=\theta^{-m}$ implies that the directional derivative $D_\alpha f(\alpha)d$ equals 0. This contradicts the fact that $D_\alpha f(\alpha)$ is nonsingular. Consequently, every regular equilibrium has a neighbourhood in which there are no other equilibria. This, together with the compactness of the simplex and the continuity of the equilibrium conditions, implies that, if $\theta^{-m}$ is a regular value, then there are only finitely many corresponding equilibria. Because of the boundary condition (c), it follows from the index theory in Proposition 1 that the number of equilibria is odd. (See, for example, Varian [27].)

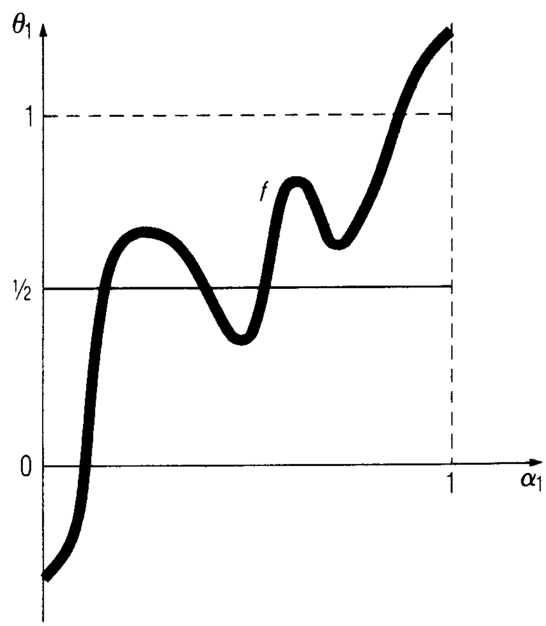


FIG. 1. The equilibrium map for a two-person economy.

In the two-person case illustrated in Fig. 1, the horizontal line $\theta_1 = \frac{1}{2}$ illustrates a typical regular endowment. The slope of $f$ does not vanish where this line is crossed. Consequently, there are finitely many equilibria, and since both the far-left and the far-right crossings of the line come from below by the boundary condition, it is clear that the number of equilibria is odd.

Fig. 1 illustrates why indeterminacy could arise for a measure-zero set of initial values for $\theta_1$. Nothing rules out the possibility that $f$ is constant over some interval of $\alpha$'s. If the economy happened to start with a value for $\theta_1$ equal to the value that $f$ takes in this range, then there would be an infinite number of nearby solutions to the equations $f(\alpha) = \theta_1$. It is intuitively clear, however, that almost all the possible choices of $\theta$ do not correspond to a flat in $f$.

This intuition is made precise by Sard's theorem, which says that if a function $f$ that maps an open subset of $\mathbf{R}^{m-1}$ (here, the interior of the simplex for $\alpha$) to $\mathbf{R}^{m-1}$ is continuously differentiable, then the set of regular values $\theta^{-m}$ has full measure. In other words, for almost all $\theta^{-m}$, $D_\alpha f(\alpha)$ is nonsingular for every value of $\alpha$ with $f(\alpha) = \theta^{-m}$. In the example shown in the figure, $D_\alpha f(\alpha)$ is nonsingular if $f'(\alpha) \neq 0$.

Since the set of regular endowments $\theta^{-m}$ has full measure for each fixed $k_0$ and $\phi$, it follows from Fubini's theorem that the set of all $\theta$, $\phi$, and $k_0$ for which $\theta^{-m}$ is a regular value also has full measure. Our arguments can be summarized in the following result:

PROPOSITION 2. *Assumptions* 1 *and* 2 *imply that, for almost all* $\theta$, $\phi$, *and* $k_0$, *there is a finite* (*odd*) *number of equilibria. Furthermore, the equilibrium weights* $\alpha$ *vary continuously with* $\theta$, $\phi$, *and* $k_0$ *in some neighborhood of each equilibrium.*

Moreover, since $D_\alpha f(\alpha)$ is nonsingular at each such equilibrium, the implicit function theorem allows us to solve locally for $\alpha$ as a function of $\theta$, $\phi$, and $k_0$ to do comparative statics.

## 6. The Multisector Model

To what extent does the determinacy of equilibrium depend on the special features of the one-sector growth model? Although the notation and assumptions can be stated more simply in the one-sector case, the reduction of finding equilibrium in a production economy with finitely many consumers to finding zeros of finitely many savings functions requires only that markets be complete and that the first and second welfare theorems hold. This, however, is true in much more general settings. Moreover, the conclusion that equilibria are determinate for generic endowments is based only on the fact that the savings function is $C^1$. The importance of the one-sector assumptions has been their use in proving the stronger proposition that the value function is $C^2$. Consequently, it is sufficient for the study of determinacy in a production economy to study that $C^2$ differentiability of the value function for that technology.

Is the value function $C^2$ is a multisector model? Araujo and Scheinkman [1] show that if the technology itself is sufficiently differentiable, a $C^2$ value function follows from a global turnpike theorem. In other words, if all socially optimal paths converge to a unique stationary state (which may, however, depend on the welfare weights) and if suitable technical conditions are satisfied, then equilibria are generically determinate. This observation is indeed the basis of the result in the one-sector case. In a multisector model, a global turnpike theorem generally requires that the discount factor be close to one (see McKenzie [20, 21]).

In contrast, if the discount factor is small and if suitable technical conditions are satisfied, then the value function is $C^2$ (see Boldrin and Montrucchio [8]). This is true despite the fact that Boldrin and Montrucchio [6] and Deneckere and Pelikan [11] have shown that with the same small discount factors, not only may optimal paths cycle but they may be chaotic. In other words, a $C^2$ value function (and determinacy) does not require a global turnpike theorem.

This leaves one important gap: There is an intermediate range of discount factors for which it has not been shown that the value function is $C^2$. Moreover, there are no known counterexamples that do not violate conditions (such as strong concavity) and that are part of the sufficient conditions for $C^2$ differentiability of the value function with a finite horizon.

Given this gap, it is important to ask, To what extent is $C^2$ differentiability of the value function actually needed? Is $C^1$ differentiability of the savings function needed? Certainly, if the savings function is $C^0$, robust examples of indeterminacy exist (see Kehoe, Levine, Mas-Colell, and Zame [18]). If, however, the savings function is Lipschitz-continuous in the utility weights $\alpha$, then, as Santos [26] has shown, this suffices for determinacy. His argument is that of the previous section: Sard's theorem requires only Lipschitz continuity. Moreover, Montrucchio [22] has shown that the value function is $C^1$ Lipschitz in initial capital stocks under relatively general conditions. Although this result is promising, the remaining gap is to show that the value function is also $C^1$ Lipschitz with respect to parameters such as $\alpha$. Again, neither a positive result nor a counterexample is available.

These problems of differentiability of the savings functions can occur even in a simple one-sector model if the consumption good and the investment good are not perfect substitutes in production. Recall that the proof of Proposition 1 depends crucially on the property of $v$ that $\partial^2 v/\partial k_t\, \partial k_{t+1} > 0$. Joint production can cause this condition to be violated. If we could ensure that $\partial^2 v/\partial k_t\, \partial k_{t+1} < 0$ everywhere, then the proof would still hold. If, however, $\partial^2 v/\partial k_t\, \partial k_{t+1}$ can change signs, then the analysis of differentiability in this one-sector model runs into the same sort of problems as in the multisector model.

*Note added in proof.* Since the final revision of this paper was submitted, important contributions to proving the $C^2$ differentiability of the value function have been made by A. Araujo ("On the Differentiability of the Policy Function") and M. S. Santos ("Differentiability and Comparative Analysis in Discrete-Time Infinite-Horizon Optimization Problems"). In particular, Santos shows that, under certain interiority conditions on the solution and differentiability conditions on the objective function, solutions to optimization problems of the sort considered in this paper vary smoothly with respect to both initial conditions and parameters. His results seem to resolve the differentiability questions for the multisector model discussed in Section 6.

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