Introduction to the Special Issue in Honor of Lloyd Shapley: Seven Topics in Game Theory[1]

by David K. Levine

Lloyd Shapley was my colleague at UCLA for many years; by the time he moved from RAND to UCLA he was already a legend – although he never acted the part doing his best to maintain the reputation of mathematicians as somewhat idiosyncratic souls. He claimed to me at one time that the optimal length of the day was 25 hours and that therefore one should wake up an hour later every day – although I do not believe he fully implemented this.

The purpose of this special issue is to celebrate Lloyd Shapley the legend. We also remember Lloyd Shapley the man – and we begin the special issue with a remembrance section collecting together some recollections and facts about Lloyd's career. The bulk of the issue is devoted to current research that advances the Shapley research agenda. Our first contribution is Roberto Serrano's "Annotated Bibliography." We believe this is the most complete guide to Lloyd Shapley's work ever compiled. Lloyd was famous, among other things, for publishing papers – significant ones – in obscure places, and in some cases not publishing them at all. Roberto and his research assistants have made a heroic effort to track it all down and tell us what it is about.

In comparison to some Lloyd Shapley was not a prolific writer. Over a career spanning sixty-three years his Google h-index is 56 – by contrast a prolific writer such as Jean Tirole over thirty-seven years has an h-index of 127. I find looking over the fine papers we have selected that they organize themselves into seven topics; and as a result I am going to focus on just seven of his papers. The striking fact is that each of these papers has led to a substantial literature of its own including the current research reported in this volume. Three of these remarkable contributions are prior to 1960 and all but one prior to 1970. The remaining contribution – and his third most highly cited paper – was written in 1996 showing that Lloyd never did get old.

1. The Shapley Value: Shapley, L. S. (1953). "A value for n-person games." *Contributions to the Theory of Games*, 2(28), 307-317.

Or as Lloyd called it: "the value." No – I'm not going to tell you what the Shapley value is – if you have to ask you shouldn't be reading this journal – and anyway Wikipedia has a perfectly usable definition. There are 11 theoretical articles in this issue advancing the idea of the Shapley value some sixty-four years after Lloyd introduced it. Perhaps less well known is the enormous impact that the Shapley value has had on applied work – and in other disciplines such as accounting – and we have an additional five articles examining applications of the Shapley value.

The theoretical 11.

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1 I am particularly grateful to Ehud Kalai, Peter Shapley, and Jennifer Byrd. Without Jennifer's support this volume never would have happened.

i. “Values for Cooperative Games over Graphs and Games with Inadmissable Coalitions” Ziv Hellman and Ron Peretz
The value, like many cooperative equilibrium notions, assumes that all coalitions are equally likely. Here the value is generalized to incorporate restrictions on the coalitions that can form that arise from the theory of social networks.

ii. “Decomposition of solutions and the Shapley value decomposition” André Casajus and Frank Huettner
There are many axiomatic characterizations of the value: indeed part of its strength lies in the robustness and the connections these axiomatic systems bring with other solution concepts. Here an axiomatic system of decomposition is introduced that connects the Shapley value with the naïve solution measuring the marginal contribution of a player to the game.

iii. “Values for Environments with Externalities – The Average Approach” Ines Macho-Stadler, David Pérez-Castrillo and David Wettstein
By measuring the worth of a coalition as a weighted average as its worth over different partitions of the players different notions of the value can be extended to characteristic function form games. Five such extensions are considered and partially characterized.

iv. “The Stochastic Shapley Value for Coalitional Games with Externalities” Oskar Skibski, Tomasz Pawel Michalak and Michael Wooldridge
This paper also attacks the issue of externalities, albeit along a different dimension. Here the problem is analyzed viewing coalition formation as a stochastic process. It is shown that the Chinese Restaurant Process leads to a value with desirable properties.

v. “Balanced externalities and the Shapley value” Benjamin McQuillin and Robert Sugden
More on externalities: here a characterization of the Shapley value for a useful class of games using an extension of the idea of threat points from bargaining is given.

vi. “The proportional Shapley value and applications” Sylvain Béal, Eric Rémila, Philippe Solal and Sylvain Ferrières
The value uses linear weights. Here the theory is extended to endogenous non-linear weights. The resulting proportional Shapley value is characterized and shown that to proportionally distribute the Harsanyi dividend.

vii. “The balanced contributions property for equal contributors” Koji Yokote, Yukihiko Funaki and Takumi Kongo
The value here is extended in another new direction with an axiom requiring balanced contributions only from pairs of players who contribute the same amount to the grand coalition. This leads to the new concept of r-egalitarian Shapley values that organizes many existing variants of the value.

viii. “Membership Separability: A New Axiomatization of the Shapley Value” Geoffroy de Clippel
Here the traditional value axioms are reexamined and strengthened: in particular additivity can be replaced by a weaker accounting property.

ix. “Is Shapley Cost Sharing Optimal?” Shahar Dobzinski, Aranyak Mehta, Tim Roughgarden and Mukund Sundararajan

Here the implications of the value for mechanism design is analyzed. The budget balance cost sharing problem is studied with the objective of minimizing worst case inefficiency. The overall conclusion is that for a class of important problems the value does a good job.

x. “Games of Threats” Abraham Neyman and Elon Kohlberg
In a game of threats the failure to reach agreement need not lead to zero value as is the case in a coalitional game. Never-the-less there is an analog of the Shapley axioms that determine a unique value and has properties similar to those of the value for coalitional games.

xi. “The Shapley Value of conjunctive-restricted Games” Jean Derks
Conjunctive-restricted games allow situations such as hierarchies where a subset of players controls a larger set. Here Shapley’s methods are extended to study these types of games.

The applied five:

i. “Fair Representation and a Linear Shapley Rule” Sascha Kurz, Nicola Maaser and Stefan Napel
The Penrose square root rule is a benchmark for “fair representation.” When there are intervals of alternatives with single-peaked preferences and a positive correlation of local voters the Penrose rule is shown to be defective and should be replaced with a linear Shapley rule.

ii. “A note on the Shapley value for airport cost pooling game” Dongshuang Hou, Hao Sun, Panfei Sun and Theo Driessen
The airport cost pooling game is a generalization of the airport game. Here it is shown how to compute the value. The value is then characterized using the property that for any pair of airplanes from different airlines the loss of withdrawal is the same for each.

iii. “Shapley Value Based Pricing for Auctions and Exchanges” Luke Lindsay
In order to apply the value to auctions and exchanges the standard value is modified so that the losers do not pay. This leads to a new set of rules for multiple item auctions which is compared to existing solutions.

iv. “Polluted River Problems and Games with a Permission Structure” Rene van den Brink, Simin He and Jia-Ping Huang
This deals with externalities in a more applied setting: how should the costs for cleaning a river be shared among the regions through which it passes? Attacking this problem using the value leads to new insights including the Upstream Limited Sharing method.

v. “Valuing Inputs Under Supply Uncertainty: The Bayesian Shapley Value” Roland Pongou and Jean-Baptiste Tondji
The problem of valuing inputs in an uncertain production environment is examined using axiomatic methods. This leads to the a priori Shapley value or the Bayesian Shapley value depending on the information structure. The theory is illustrated with an application to fidelity networks.

2. The Assignment Problem: Gale, David, and Lloyd S. Shapley. “College admissions and the stability of marriage.” *The American Mathematical Monthly* 69.1 (1962): 9-15.
This is of course Gale and Shapley – some of the unique story behind the writing can be found in the remembrances. Again – I’m not going to tell you what the matching problem is – this time

rather than Wikipedia I will refer you to the Nobel citation. An additional eight articles advance Lloyd's work on the assignment problem.

i. "From the bankruptcy problem and its Concede-and-Divide solution to the assignment problem and its Fair Division solution" Christian Trudeau
Here the equivalence between sharing methods is studied.

ii. "The Stochastic Stability of Decentralized Matching on a Graph matching" Leonardo Boncinelli and Paolo Pin
While here an evolutionary approach is taken to matching.

iii. "The Stable Fixtures Problem with Payments" Peter Biro, Walter Kern, Daniel Paulusma and Peter Wojuteczky
We continue here with matching on graphs. This contribution furthers the study of multiple partner matching games. Algorithmic convergence time is given.

iv. "Competitive Equilibria in School Assignment" Umut Dur and Thayer Morrill
Shapley and Scarf developed top trading cycles to solve the competitive equilibrium problem of a market where buyers are matched to homes. Here the idea is extended to the school assignment problem.

(have not received final paper yet) v. "Lone wolves in infinite, discrete matching markets" Ravi Jagadeesan
The problem of moving from finite to infinite matching markets is studied. A key fact that the unmatched agents are the same in all stable outcomes fails – but never-the-less the deferred acceptance mechanism is often strategy proof.

(connected to missing paper) vi "Strategy-Proofness of Worker-Optimal Matching with Continuously Transferable Utility" Ravi Jagadeesan, Duke Kominers and Ross Rheingans-Yoo
As the title says: the proof uses the new "lone wolf" theorem from the previous contribution.

vii. "Limited Choice in College Admissions: An Experimental Study" Wei-Cheng Chen, Yi-Yi Chen and Yi-Cheng Kao
The matching literature has had an enormous impact on applied research. Consequently it is much studied in the laboratory and empirically. Here different institutions for limiting students' choice sets are studied in the laboratory. The implications of limiting multiple applications – and possible welfare improvements are shown.

vii. "Suboptimal Behavior in Strategy-Proof Mechanisms: Evidence from the Residency Match" Alex Rees-Jones
Our final contribution to matching is an empirical study. Data from the medical residency matching is used – and it seems that "students pursue futile attempts at strategic misrepresentation."

3. Potential Games: Monderer, Dov, and Lloyd S. Shapley. "Potential games." *Games and Economic Behavior* 14.1 (1996): 124-143.
I do have this in the right place: this is Lloyd's third most highly cited work. (Lloyd has a Google Scholar page of his own.) It was published right here in *Games and Economic Behavior* – although Ehud Kalai tells me he was inclined to reject it on the grounds he had no idea what use it might have. Equilibria in potential games have special properties – not least of which is that they have strong stability properties with respect to learning. This is a "Lloyd could see the future where the

rest of us could not" example: these games are useful because there are a surprising lot of them although you would not think so from the definition. I am pleased that we have a few recent contributions to the literature on potential games in this volume.

i. "A universal construction generating potential games" Nikolai S. Kukushkin
One problem with potential games is figuring out which games have potentials. Here a novel method is introduced extending the idea that congestion games have potentials to a broader class of games with structured utilities.

ii. "Nonatomic Potential Games: the Continuous Strategy Case" Man Wah Cheung and Ratul Lahkar
Large games – either large populations or large strategy sets or both – are of tremendous importance in economics. Hence the non-atomic model of agents and continuous strategy spaces. This contribution shows how potentials can be used to analyze these large games.

4. Power Index: Shapley, L.S., Shubik, M., 1954. "A method for evaluating the distribution of power in a committee system." *Amer. Polit. Sci. Rev.* 48, 787–792
The Banzhaf power index was invented by Lionel Penrose in 1946: it measures the power of a voter by the probability of being the decisive voter in a winning coalition. This anticipates some of the key ideas of the Shapley value and Shapley and Shubik subsequently developed a power index more consonant with the value and studied it and the Banzhaf index in a series of papers. Three of our contributions advance the study of Banzhaf and Shapley-Shubik power indices.

i. "Effectivity and Power" Dominik Karos and Hans Peters
An important theme of this volume is the extension of existing methods – generally those developed by Shapley – to a broader class of games. This paper is in that vein – by extending the class of games for which power indices are defined it becomes possible to apply the theory to examples such as that of the U.S. legislative system that previously defied analysis.

(haven't received final paper yet)ii. "The Conditional Shapley-Shubik measure for Ternary Voting Games" Jane Friedman and Cameron Parker
Continuing to extend the power index idea, this contribution studied games where in addition to voting yes or no it is possible to abstain.

iii. "The Axiom of Equivalence to Individual Power and the Banzhaf Index" Ori Haimanko
Here the axiomatic approach to power indices is applied – and used to give a new characterization of the Banzhaf power index

4. Learning in Games: Shapley, Lloyd S. "Some topics in two-person games." *Advances in Game Theory* 52.1-29 (1964): 1-2.
You knew I was coming to this one, right? You probably would not guess from the rather bland title that one of the topics in two-person games is an extremely robust example of a simple learning procedure (fictitious play) that converges to a cycle – it is in fact the famous game of rock-paper-scissors. This has played a key role in my own work with Drew Fudenberg on learning in games, and continues to play an important role in our understanding of the convergence and divergence of learning processes. Two of the contributions here advance the agenda of studying convergence and divergence of learning processes.

i. "The Query Complexity of Correlated Equilibria" Sergiu Hart and Noam Nisan
The classical Shapley problem of cycles is avoided in modern learning theory by stochastic dynamics. The key point here is that this is in fact necessary.

ii. “Approachability of convex sets in generalized quitting games” János Flesch, Rida Laraki and Vianney Perchet
The key technical tool in studying the stochastic dynamics and showing that they avoid the classical Shapley problem is Blackwell approachability. Here we find simple geometric conditions for approachability in an important class of games.

5. Market Games: Shapley, Lloyd S., and Martin Shubik. "On market games." *Journal of Economic Theory* 1.1 (1969): 9-25.
You probably associate the study of market games with Martin Shubik rather than Lloyd Shapley, but in the early years the two of them did extensive joint work on these games. The topic is still going strong: we have four papers on the topic in this issue, two of them co-authored by Martin Shubik.

i. “Money as Minimal Complexity” Pradeep Dubey, Siddhartha Sahi and Martin Shubik
A graph theoretical approach to markets is taken and three key graphs are identified: the star, the cycle and the complete graphs. The star mechanism stands out because there is a single money used in every transaction and this mechanism has a minimality property.

ii. “Graphical Exchange Mechanisms” Pradeep Dubey, Siddhartha Sahi and Martin Shubik
This paper is the theoretical underpinning of “Money as Minimal Complexity” laying out carefully what graphical exchange mechanisms are and the definition of minimal complexity.

iii. “Shapley’s Conjecture on the Cores of Abstract Market Games” Zhigang Cao, Chengzhong Qin and Xiao-guang Yang
I’m going to quote the abstract here as it does a particularly good job of explaining this paper: “Shapley (1955) introduced the model of an abstract market game as a generalization of the assignment game model, among several other models. He conjectured that abstract market games possess non-empty cores. We analyze properties of abstract market games and provide a proof of this conjecture for cases with four or fewer players. We show by example that, in general, the structure of an abstract market game is not strong enough to guarantee the nonemptiness of the core. We establish supplemental conditions for the conjecture to hold. Our supplemental conditions are satisfied by the assignment games and abstract market games with one side consisting of a single player as with package auction games in Ausubel and Milgrom (2002).”

iv. “Noncooperative Oligopoly in Markets with a Continuum of Traders and a Strongly Connected Set of Commodities” Francesca Busetto, Giulio Codognato, Sayantan Ghosal, Ludovic Julien and Simone Tonin
Here the Shapley trading window model is studied where there are both large and small traders: a significant weakening in the conditions for existence of equilibrium can be established.

6. Stochastic Games: Shapley, Lloyd S. “Stochastic games.” *Proceedings of the National Academy of Sciences* 39.10 (1953): 1095-1100.
Yes, he invented these as well. We have two papers studying zero-sum stochastic games – although in recent years the study of non-zero sum stochastic games has become a thing.

i. “Tauberian theorems for general iterations of operators” Bruno Ziliotto

Tauberian theorems assert that the finite time averages converge uniformly if and only if the discounted payoffs converge uniformly as the discount factor approaches one. Here several theorems are proven and applied to the study of values and strategies in zero sum stochastic games.

ii. “Zero-sum Revision Games” Fabien Gensbittel, Stefano Lovo, Jérôme Renault and Tristan Tomala
Again a good abstract: “In zero-sum asynchronous revision games, players revise their actions only at exogenous random times. Players’ revision times follow Poisson processes, independent across players. Payoffs are obtained only at the deadline by implementing the last prepared actions in the ‘component game’. We characterize the value of this game as the unique solution of an ordinary differential equation.” The results are then applied to analyze particular classes of games.

iii. “Acceptable Strategy Profiles in Stochastic Games” Eilon Solan
Also known as “satisficing meets Lloyd Shapley” this one is not for zero sum games. Here players have utility targets and conditions under which simple strategies enable them all to achieve high targets are given.

7. The Core: L.S. Shapley, M. Shubik. “The core of an economy with nonconvex preferences,” Rand Corporation, 1963
He was one of the greatest game theorists of all time: of course he wrote about the core. Here are two current papers advancing that research.

i. “On a class of vertices of the core” Michel Grabisch and Peter Sudhölter
This advances Lloyd’s geometrical methods and shows that there are balanced games whose core has vertices which are not min–max vertices if and only if there are at least four players.

ii. “Competitive Pricing and the Core: with Special Reference to Matching” Joseph Ostroy
Here the idea of subdifferentiability is used to unify Shapley’s contributions to the theory of the core with his work on matching.

8. Additional Topics:
While Lloyd was not prolific he did contribute in many areas. A few of our contributions do not fit into the “big” Shapley topics but are never-the-less significant papers in their own right.

Shapley, L. S. (1963). Solutions of compound simple games. *Advances in Game Theory*, (52), 267-280.
Games are simple if coalitions either win or lose and these can be combined into larger games.

Shapley, Lloyd S., and Martin Shubik. "Pure competition, coalitional power, and fair division." *International Economic Review* 10.3 (1969): 337-362.

“Fair Stable Sets of Simple Games” Eduard Talamas
This contribution advances both the work on simple games and on fair division, describing how to construct fair stable sets for compound simple games.

Our final contributions are even more eclectic.

“Coalition Preferences with Individual Prospects” Manel Baucells and Dov Samet

Earler work by Baucells and Shapley (2008) gave conditions under which coalitions have complete preferences. This finishes that work by providing simpler and weaker conditions.

“Three Little-Known and yet Still Significant Contributions of Lloyd Shapley” Jingang Zhao
This paper tracks down the history of some earlier contributions of Lloyd’s.

“Lloyd Shapley and Chess with Imperfect Information” Alexander Matros
Those who knew Lloyd knew his love of Kriegsspiel – chess where the opponents’ moves are hidden. To wrap up our issue we have a game theoretic study of some problems in Kriegsspiel.

Some Remembrances of Lloyd

Lloyd and Me

By Peter Shapley

There's been plenty said about Lloyd's mathematical achievements, thankfully a lot of it before his death when he could enjoy it, so I won't dwell on them.

Lloyd was my father, and father to my brother Chip. When growing up, I often was asked what my father did for a living. "He's a mathematician," I said. It seemed to me to be as normal an occupation as a doctor or an engineer or a plumber or an auto mechanic. But it wasn't.

A few years ago, Lloyd's nephew Mel mentioned that whenever he visited our house, there were puzzles – logic puzzles – and games all around. They weren't really for Chip and me, they were my father's toys, toys he had because he wanted constant logical stimulation. And that rubbed off on both of us, steering us in the scientific direction in our educations and careers.

But Lloyd wasn't just limited to pure logical exercises, he was interested in all forms of knowledge, from history and literature to geology. When my mother Marian went back to school in her fifties, eventually earning advanced degrees in linguistics, Lloyd learned enough of that field to understand what was going on, and he was able to serve as an outside member on the dissertation committee of Marian's classmate and friend – and now my wife – Feng-hsi.

And Lloyd loved music. He was an accomplished pianist, though he personally never felt he was all that good. But he enjoyed playing his old Steinway. He loved going to concerts, when he could get to them – in his later years, the local Palisades Symphony performances were a highlight, and I remember vividly how excited he was during and after the Nobel concert in Stockholm.

Besides all that, Lloyd enjoyed the outdoors. My parents' courtship was largely based in Yosemite, and from the time I was about ten, we made regular trips up there. He also did a number of campouts with Chip and me when we were in Boy Scouts. The trips to Yosemite with Marian continued after we were grown and moved out, even though the two of them were regularly traveling all over the world. I don't think, however, that he returned to Yosemite after Marian passed away. But he was very happy several years ago when I gave him a photo we had taken of my son Ricky on a hike up there.

And Lloyd was a baseball fan. He grew up a Red Sox fan, though he adopted the Dodgers when they followed him to California. Baseball, with all the statistics and strategy, appealed to him much more than other sports, and the Dodgers, as a pretty good team, made it easy to follow the game all season long, year after year. But he was thrilled when, at the age of 81, he was finally able to see his Red Sox win the World Series.

I miss talking baseball with my father, but what I really miss is doing all sorts of things with him, talking to him about anything, science or history or whatever construction project he was working on. His sense of humor, whether it was subtle wordplay or horrible puns, his knowledge of seemingly everything. From talking to his friends and colleagues and especially family over the past few months, the past couple of years, I'm not the only one who feels that way.

Lloyd wasn't given to sentiment, but his ties to his family were very strong indeed, and we all miss him.

The King and Lloyd

By Alvin E. Roth

My first long conversation with Lloyd was when I visited him in 1974, having just finished my Ph.D. dissertation. I made what felt like a pilgrimage to see him at the RAND Corporation in Santa Monica, to tell him about my work.

My last long conversations with Lloyd were in December, 2012, when we both attended the Nobel Prize celebrations in Stockholm. A careful observer might have noted that we had both aged a little since our first meeting. Lloyd walked with a cane, and that led to a small bit of logistics. When you receive the Nobel from the King of Sweden, the King presents you with two packages, containing the diploma and the medal, which you hold with your left hand so that the two of you can shake hands. Lloyd was concerned about how he would manage this with his cane, and so we agreed that I would walk with him and hold his cane while he received the packages and shook hands, and return it to him immediately afterwards. Hence the picture below.

A subsequent encounter with the King had a touch of game theory. The night after the awards ceremony (and a very big dinner party in the City Hall) there is another dinner party in the royal palace, followed by a reception. At some point Lloyd was fatigued and ready to return to the hotel. But royal protocol dictates that no one should leave before the King, and that when the King leaves the party is over, and everyone leaves. So if the King had been told that one of his guests of honor was ready to go home, he would have felt obliged to depart and end the party to make this possible. To avoid ending the party prematurely, Lloyd therefore exited through the kitchen with the help of palace staff, without passing by the King and without alerting him and disrupting the party. I thought it very appropriate that he exited with this small game-theoretic flourish.

Experimental Lloyd

By David K. Levine

I have a near infinite supply of Shapley stories – not to speak of unspeakable gratitude for the many kindnesses which he did for me. One story sticks in my mind and although I cannot verify its authenticity this is a volume about Lloyd the legend so I will tell it anyway. The story is this: in the early days of the RAND corporation a number of experiments were carried out on group decision making. The goal was to discover what was the best number of people to have in a group in order to quickly solve a difficult problem. A number of groups were organized and each was given a time limit, a mathematical problem too difficult to solve within the time limit, and the group was asked to provide their best guess of the answer. It seems that the number of people in the group made no difference – the only thing that mattered whether Lloyd was in the group. No matter how many others there were if Lloyd was their they got (more or less) the right answer – and not otherwise.

Gale and Shapley

By David K. Levine and Peter Shapley

One of Lloyd's most celebrated works is his paper with David Gale "College admissions and the stability of marriage" published in that great journal of economic theory *The American Mathematical Monthly* in 1962. The circumstances surrounding the writing of that paper are legendary – and like most legends the reality is somewhat obscure. Lloyd's son Peter Shapley has been able to locate the original correspondence between Lloyd and David and with Peter's permission we are now in a position to verify that many of the rumors are true.

Our first document is the statement of the problem sent originally from David to Lloyd. Lloyd received David's letter at noon and sent off the reply by 4:00 the same day. Our second document is that reply. It is remarkable that so short a letter should result in a Nobel prize – not to say have such an impact on our thinking and on our lives. Our third document is David's reply to Lloyd. We reproduce also Lloyd's reply – in effect happy to let David do the writing!

The paper was "reject and resubmit": according to Lloyd the paper was rejected by two referees but kept alive by a kind act of the editor. We do not, unfortunately, have the reports or editor's letter, but we do have Lloyd's letter to David Gale, our document five. Fortunately it seems they were inclined to resubmit.

We have reproduced also the final page from the second draft of the paper: a spirited defense of mathematics.

There is an old joke about mathematicians that is apropos: A mathematician is sitting in his office, working. He smells smoke. He goes out and sees a fire has started in the kitchen. He knows there's a sink and a bucket, so he knows how to put the fire out. Problem solved. So he goes back to his office to get back to his real work. Lloyd did pretty much the same thing. He solved the college admissions problem back in 1960, then went back to his real work. Problem solved. He let David write it up, and it wasn't until 30 years later that Al Roth actually applied the solution to the real world. At least the house didn't burn down in the meantime.

Lloyd

11 October 1960 L-20115

Professor David Gale
Department of Mathematics
Brown University
Providence 12, Rhode Island

Dear Dave:

A stable connubiation always exists. Here is a constructive proof. Let each boy propose to his best girl. Let each girl with several proposals reject all but her favorite, but defer acceptance until she is sure no one better will come her way. The rejected boys then propose to their next-best choices, and so on, until there are no girls with more than one suitor. Marry. The result is stable, since the extramarital liaisons that were previously rejected will be disliked by the girl partners, while all others will be disliked by the boy partners.

In this solution (which is not generally unique) every boy does at least as well as he could in any stable set of marriages. Ties in the girls' rankings could destroy this property. Reversing the roles obviously produces an analogous girls' optimum.

The above proof remains valid in the presence of a surplus of girls; a surplus of boys is handled by introducing dummies.

The admissions problem can be solved in exactly the same way. By properly scheduling the applications, invitations, rejections, etc., one can obtain both college-optimum and student-optimum solutions.

Cordially,

L. S. Shapley

LSS:mr

RAND-S.M.-REC'D 1960 OCT 25 PM 1:37

Addressee's Copy

BROWN UNIVERSITY

PROVIDENCE 12, RHODE ISLAND

ANS. BY L S Shapley
LETTER No. L-22165
DATE 4-4-60

October 20, 1960

Dr. L. S. Shapley
The RAND Corporation
1700 Main Street
Santa Monica, Calif.

SHAPLEY 33164

Dear Lloyd:

Your solution of the M. P. is lovely! Had you already solved it in the past, say in connection with assigning students to eating clubs at Princeton? I wondered, since you came up with the answer so quickly.

Although I haven't checked it through in detail, doesn't your algorithm have the property that the men have nothing to gain by falsifying their preferences, though this is not true for the women? Yes!

In the context of college admissions I thought of trying to generalize the model. Instead of the colleges rating individual students, let them express preferences among subsets of students. My only result was the following setup with no stable solution. A football player, F, a baseball player, B, and an all-round man, A, who plays a little of both, are applying to Harvard and Yale. B prefers Harvard, F and A prefer Yale. Harvard will only admit one man and rates them F, B, A. Yale would like best to get both F and B, second choice A alone. Every assignment is unstable in the sense that there will always be a coalition of applicant and college which could change the assignment with mutual advantage.

Don't you think this thing should be written up and published somewhere? It's your problem now and I don't want to steal the ball, but would be glad to collaborate if you wish, including writing a draft of something. It seems to me there are two things that could be done. Either write a mathematical diversion for, say, the Monthly, or make a moderately serious proposal in connection with college admissions. (Perhaps you read the recent New Yorker article on college admissions that started me thinking about this whole thing.) In connection with the second approach, perhaps John Williams would be interested. It seems to me it might be possible to persuade a group of colleges to use your algorithm on a dry run basis some year and then compare the classes they actually get by their own methods with what your routine would give them. What's to lose? There are, of course, complications in the realistic admission problem like allocation of scholarships, etc., but I think these might be worked into the scheme.

I'll be most interested in your reactions.

Best regards,

Dave

David Gale

DG:hrk
c.c. to J. D. Williams

Lloyd

4 November 1960 L-22165

Professor David Gale
Department of Mathematics
Brown University
Providence 12, Rhode Island

Dear Dave:

I gratefully accept your offer to write a draft of something. Frankly, without your instigation, I would do nothing at all. Be guided by your unerring sense of what is fitting. My only suggestion; why not a letter to the Department of Correction and Amplification?

As ever,

L. S. Shapley

LSS:ds

Ref. Inc. 53164

May 12, 1961 L-10072

Professor David Gale
Mathematics Division
Brown University
Providence 12, Rhode Island

Dear Dave:

I think your inclination to re-submit is sound. On the other hand, I think most of the referee's points are well taken. I am not sure that I agree that Section 6 is a waste, even from the mathematical point of view, since it contains the corollary that the final assignment is independent of sit-outs, and proposes two generalizations of the mathematical model for investigation. Also, I think that he is being over-parochial in the arguments beginning "If the purpose of the paper is..." and "if that is the purpose, then... ." It seems to me that there is nothing wrong in principle in discussing the non-mathematical implications of a mathematical result in a journal like the Monthly, especially when the application is to a field with which most of the readers have direct experience. No doubt the reviewer's real criticism is not that we did this, but that we overdid it--and he is probably right.

As before, I'll let you do the work. The above paragraph may contain verbal ammunition that you can fire at the editor. As a further contribution to the common effort, I have gone through the paper with a pruning hook seeing how much I could lop off without doing any real damage. I think the cuts total almost two pages, mostly from the first and last two sections. I don't necessarily recommend keeping Section 6 even in its pruned form. Better perhaps to junk it entirely in favor of the still shorter piece that you wrote that we didn't use, or something like it. Then, perhaps, lump it with Section 7 in a single "Addendum." I leave this up to you.

Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with "a head for figures", or that they "know a lot of formulas". At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1. The argument is carried out not in mathematical symbols but in ordinary English, there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical.

What then, to raise the old question once more, is mathematics? The answer it appears is that any argument which is carried out with sufficient precision is mathematical, or, as has been remarked not entirely facetiously, the difference between mathematicians and other people is that a mathematician is able to conceive of an argument requiring more than two steps.