Does Market Incompleteness Matter?*

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Abstract

This paper argues that incompleteness of intertemporal financial markets has little effect (on welfare, prices, or consumptions) in an economy with a single consumption good, provided that traders are long-lived and patient, a riskless bond is traded, shocks are transitory, and there is no aggregate risk. In an economy with aggregate risk, a similar conclusion holds, provided traders share the same CRRA utility function and the right assets are traded. Examples demonstrate that these conclusions need not hold if the wrong assets are traded or if the economy has multiple consumption goods.

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1 Introduction

Does market incompleteness matter — for welfare, for prices, for consumption? If the time horizon is short, market incompleteness generally will matter because it entails the inability to insure against risk. But when the time horizon is long, intuition suggests that market incompleteness may not matter if traders can self-insure — borrowing in bad times, saving in good times. This paper presents a rigorous, theoretical look at this intuition in a general equilibrium setting. Our focus is on welfare, but our analysis has implications for prices and consumption as well.

We frame our analysis in an infinite horizon exchange economy, populated by infinitely-lived traders.[1] Intertemporal trade in our model economy is accomplished through short-lived real assets. Traders maximize discounted expected utility of their consumption stream, using a common subjective discount factor.[2] We treat traders' common discount factor as a parameter, and ask about the welfare losses of market incompleteness as the discount factor tends to 1. (Our emphasis on asymptotic behavior is in the spirit of the Folk Theorem for infinitely repeated games.) We show in Theorem A that market incompleteness will not matter (in the sense that welfare losses tend to 0 as the discount factor tends to 1), provided shocks are transitory and purely idiosyncratic (so there is no aggregate risk), and that only a single consumption good is traded. This conclusion is robust to assumptions about consumer preferences (we assume only that utility is separable over time and that the first derivatives of period utility functions are convex) and to assumptions about the asset structure (we assume only that riskless bonds are traded at each date-event; the asset structure is otherwise arbitrary).[3] On the other

[1]Our conclusions would be much the same in a finite horizon world, provided we were to treat both the length of the horizon and the discount factor of traders as parameters; the infinite horizon framework seems more natural and is definitely more convenient. Another alternative would be to consider an infinite horizon world populated by overlapping generations of finitely lived traders. Analysis of such a model would be complicated by the need to treat both the discount factor of traders and the length of their lifetimes as parameters, but we believe the conclusions would be similar to those obtained here.

[2]Because Ponzi schemes must be ruled out, the definition and existence of equilibrium are subtle issues. We rely here on Levine and Zame (1996); Magill and Quinzii (1994) provide an equivalent formulation.

[3]Much of the literature assumes a specific asset structure; typically riskless bonds only, or equity only, or riskless bonds plus equity only. The fact that we allow for arbitrary asset structures seems important to us, since adding assets may be Pareto worsening.

hand, this conclusion is fragile to each of the other assumptions. In particular, we show in Examples 1 and 2 that market incompleteness may matter when there is aggregate risk (even when other stringent conditions are met), and we show in Example 3 that market incompleteness may matter when there is more than one consumption good. (Constantinides and Duffie (1996) have shown that market incompleteness may matter if shocks are permanent.) Market incompleteness matters in Examples 1 and 2 because aggregate risk affects prices; in particular, it drives up the riskless interest rate, making borrowing constraints tighter. Market incompleteness matters in Example 3 because relative price effects provide an additional, untraded source of risk.[4] Finally, we provide in Theorem B some (strong) sufficient conditions on preferences, endowments and the asset structure in order that market incompleteness not matter in a one-good exchange economy with both idiosyncratic and aggregate risk; a crucial condition is that the market for aggregate risk is complete.

Of course, the idea that patient consumers can self-insure is not a new one. A familiar partial equilibrium expression of this idea is due to Yaari (1976). Yaari analyzes the optimal lifetime consumption pattern of a perfectly patient trader who lives a long (finite) lifetime, faces an i.i.d. endowment stream, and can borrow and save risklessly at a 0 interest rate. Yaari shows that, as the trader's lifetime tends to infinity, the optimal consumption plan converges to constant average consumption and the (per period average) utility of the optimal consumption plan converges to the utility of constant average consumption. Our Theorem A parallels Yaari's work (and rests, in a similar way, on the Law of Large Numbers), but there are a number of important differences. First, and most importantly, Yaari treats a one-consumer optimization problem, while we treat an equilibrium problem. Second, Yaari allows consumption to be negative, while we require consumption to be non-negative. Thus Yaari's consumers do not face borrowing constraints, while ours do. As we shall see, these borrowing constraints play an important role in our analysis.[5] Third, Yaari *assumes* that the riskless interest rate is 0, while we *derive* the riskless interest rate. In our context, the riskless interest rate cannot be much above 0, but might be much below 0. This has important consequences for the form of the alternative plans we use to provide lower bounds on equilibrium utilities. Because interest rates are different from 0, these plans must maintain a

[4]To insure against price risk in such a setting, it would appear that traders would require access to assets whose dividends depend on *prices*.

[5]The possibility of negative consumption is crucial to Yaari's conclusion that the consumer's optimal consumption plan converges to constant average consumption.

delicate balance between consumption smoothing and the growth of debt. Because interest rates might be negative, these plans call for consumers to borrow when necessary and repay when possible — but not to save. (Keep in mind that we are discussing alternative feasible plans, not equilibrium plans. At equilibrium, of course, market clearing means that when some traders borrow others must save.)

The implications of market incompleteness have been the subject of much interest in the macroeconomics/finance literature, but the focus there has been on prices, rather than on welfare. Much of the motivation for this literature has come from the observation, following Mehra and Prescott (1985), that the standard Lucas (1978) asset pricing model has a great deal of trouble explaining the observed high rates of return on equities (the "equity premium puzzle") and the low rates of return on riskless securities (the "riskless rate puzzle"). Most of this literature, of which Telmer (1993), Lucas (1994), and Heaton and Lucas (1996) are representative, provides numerical solutions to models calibrated to observed parameters. (Kocherlakota (1996) provides an excellent and intuitive discussion of the numerical findings.) This literature generally concludes that market incompleteness alone is not sufficient to explain the quantitative features of the data. Heaton and Lucas (1996), in particular, argue that explaining the data requires substantial trading frictions, persistence of idiosyncratic shocks, or correlation between idiosyncratic and aggregate shocks — in addition to market incompleteness. Because we focus on exchange economies, on welfare, and on the asymptotic limit as the discount factor tends to 1, our work is not precisely comparable to the macroeconomics/finance literature, but it is probably fair to say that our results point in a different direction. In particular, we find that, in the presence of aggregate risk, market incompleteness alone may have substantial effects. We stress that we provide theorems, not numerical solutions; on the other hand, we do not provide quantitative results.

Calvet (1997) and Willen (1999) have examined some similar questions in a particular incomplete markets framework (one good, exponential utility, normally distributed asset returns, negative consumption allowed). Calvet shows that incompleteness matters a great deal — in particular, volatility of asset prices may be extremely high — if consumers are quite impatient. Willen, on the other hand, shows that incompleteness matters very little if consumers are reasonably patient.

Our results provide a theoretical echo to the empirical conclusions of Townsend (1992) concerning village economies: a great deal of risk sharing may take place even in the absence of a complicated structure of financial instruments.

The questions we raise here are reminiscent of what Friedman (1957) called the *permanent income hypothesis*: that traders behave so as to maintain a constant marginal utility of income. Friedman's discussion of the permanent income hypothesis was informal; he did not offer any specific formalization. Yaari (1976) established the validity in his setting of a one-consumer formalization: "... as the number of planning periods becomes very large, optimal consumption tends to permanent income ..." Since Yaari's results and our Theorem A lead to the same conclusions about utility, we here establish the validity, in the setting of Theorem A, of an equilibrium formalization: as consumer discount factors tend to 1, the utility of equilibrium consumption tends to the utility of permanent income. In the setting of Theorem B, however, the utility of equilibrium consumption does *not* tend to the utility of permanent income, but rather to the utility of permanent share. In our various examples for which market incompleteness matters, consumption is not perfectly smoothed, and again the utility of equilibrium consumption does *not* tend to the utility of permanent income. Bewley (1977) provides a rather different formalization of the permanent income hypothesis. Bewley's framework is different from Yaari's in that Bewley treats an equilibrium formulation rather than a one-consumer formulation, and different from ours in that Bewley's model economy is populated by a continuum of traders who are *ex ante* identical but subject to idiosyncratic shocks, while our model economies are populated by heterogeneous traders. It is probably fair to say that our work suggests that the permanent income hypothesis, in the traditional sense, is not likely to hold, except in a world that can be approximated by a one-good world with no aggregate risk

Despite a superficial similarity, our work is quite different from the body of work in the finance literature showing that, when information is revealed gradually, frequent trading of long-lived assets may lead to dynamically complete financial markets and hence to efficiency. (See Kreps (1981) and Duffie and Huang (1987) for instance.) In our framework, information is not revealed gradually, only short-lived assets are traded, trading is not frequent, and markets are not dynamically complete.

2 The Economy

2.1 Time and Uncertainty

Time and uncertainty are represented by a countably infinite tree S. Each node on the tree represents a date-event. The initial date-event (the root of the tree) is denoted by $0 \in S$. For date-events $s, s' \in S$, we write $s \leq s'$ to mean that s' follows s (and s precedes s'). For each date-event $s \in S$ other than 0, we write s^- for the (unique) date-event that immediately precedes s, s^+ for the set of date-events that immediately follow s, $s^{+2} = (s^+)^+$ for the set of date-events that follow date-events that immediately follow s, and so forth.

Each $s \in S$ is a finite history of exogenous events; we denote the length of that history by $t(s)$. Thus $t(0) = 0$ and $t(s^-) = t(s) - 1$. A complete path through the tree S is a complete history of exogenous events; write $\mathcal{H}$ for the set of all such infinite histories. Given a finite history $s \in S$ with $t(s) \geq t$ (respectively, an infinite history $h \in \mathcal{H}$) and a date t, write s_t (respectively, h_t) for the history up to and including time t. Thus $s_t \in S$ and $t(s_t) = t$ (respectively, $h_t \in S$ and $t(h_t) = t$).

We assume that exogenous events follow a finite Markov chain with state space Ω and strictly positive transition probabilities.[6] That is, there is a map $s \mapsto \omega_s : S \to \Omega$ which is a bijection on the set s^+ of immediate successors of every node $s \in S$. For $s \in S$ and $\sigma \in s^+$, $\pi(\sigma|s) = \pi(\omega_\sigma|\omega_s)$ is the conditional probability that date-event σ occurs, given that s has occured. Because the underlying Markov chain is finite and has strictly positive transition probabilities, these conditional probabilities $\pi(\sigma|s)$ are strictly positive and uniformly bounded away from 0 and 1. Note that $\sum_{\sigma \in s^+} \pi(\sigma|s) = 1$. For $s \in S$, write π_s for the unconditional probability that the date-event s is reached. Because some date-event is reached at every date, $\sum_{t(s)=t} \pi_s = 1$ for every $t \geq 0$.

2.2 Commodities

There are $L \geq 1$ commodities available for consumption at each date-event. The *commodity space* is the space $\ell^\infty(S)^L$ of bounded functions $x : S \to \mathbf{R}^L$. For $x \in \ell^\infty(S)^L$, we write $x_s \in \mathbf{R}^L$ for the bundle specified at date event s. A *con-*

[6] It would suffice that the underlying Markov chain be recurrent. We assume strictly positive transition probabilities for convenience.

sumption plan is an element of $\ell^\infty(S)^L_+$; that is, a bounded function $x : S \to \mathbf{R}^L_+$. Commodities are traded only on spot markets (there are no markets for contingent commodities), so spot prices are (not necessarily bounded) functions $p : S \to \mathbf{R}^L_+$. Given a spot price p, we write $p_s \in \mathbf{R}^L$ for the spot prices at date-event s. It is convenient to take the first good as numeraire, and to normalize so that the spot price of the numeraire good is 1 at each date-event $s \in S$.

2.3 Securities

Intertemporal trade takes place through the exchange of securities (assets). For simplicity, we assume that J securities are available at each date-event, that security returns are denominated in units of the numeraire commodity, and that each security is *short-lived*, yielding returns only at the immediate successor nodes. Security A_j traded at the date-event s yields $A_j(\sigma)$ units of the numeraire good at the date-event $\sigma \in s^+$; the portfolio $\theta \in \mathbf{R}^J$ of securities at date-event s yields dividends of $\mathrm{div}_\sigma \theta = \sum_j \theta_j A_j(\sigma)$ units of the numeraire commodity at the date-event $\sigma \in s^+$. (Note that $\mathrm{div}_\sigma : \mathbf{R}^J \to \mathbf{R}$ is a linear operator.) We assume that for each s there is a portfolio ψ such that $\mathrm{div}_\sigma \psi > 0$ for each $\sigma \in s^+$; this will certainly be the case if a riskless bond is traded at each date-event. Security prices are functions $q : S \to \mathbf{R}^J$; we write $q_s \in \mathbf{R}^J$ for security prices at date-event s.

2.4 Utilities

There are N infinitely lived traders $i = 1, \ldots, N$, having utility functions $U^i : \ell^\infty(S)^L_+ \to \mathbf{R}$. We assume traders maximize the discounted sum of expected utility, according to a stationary period utility function u^i. Thus

$$U^i_\delta(x) = (1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{t(s)=t} \pi_s u^i(x_s).$$

We assume that u^i is a smooth (C^2) strictly concave, strictly increasing function.[7] We write U^i_δ in order to emphasize the dependence on the discount factor δ, which we think of as a parameter. The leading factor $(1-\delta)$ normalizes so that the discounted utility of the constant consumption stream c is $u^i(c)$, independent of the discount factor δ.

[7]Note that utility functions are bounded below.

2.5 Endowments

We assume endowments are Markov (i.e., the endowment e^i_s of trader i at date-event s depends only on the state ω_s of the underlying Markov chain at s) and bounded away from 0 (i.e., there is a constant $m > 0$ such that $e^i_s \geq (m, \ldots, m)$ for each i, s).

2.6 Budget Sets and Debt Constraints

Given spot prices p and security prices q, trader i chooses a consumption plan $x^i : S \to \mathbf{R}^L_+$ and a portfolio trading plan $\theta^i : S \to \mathbf{R}^J$. At each date-event s, trader i faces a spot budget constraint which may be written:

$$p_s \cdot x^i_s + q_s \cdot \theta^i_s \leq p_s \cdot e^i_s + \mathrm{div}_s \theta^i_{s^-} \tag{1}$$

That is, expenditure to purchase consumption and to purchase securities does not exceed income from sale of endowment and from dividends on securities acquired at the previous date-event. (Recall that securities are denominated in units of the numeraire commodity, whose price is normalized to 1.)

In our infinite horizon setting, these spot constraints are not sufficient to rule out Ponzi schemes (doubling strategies) and hence unlimited amounts of borrowing. As we show in Levine and Zame (1996), the additional constraints necessary to rule out Ponzi schemes may be formalized in any of a number of ways, each of which leads to an equivalent notion of equilibrium.[8] Here we find it convenient to formalize the constraints by requiring that it should be possible to repay *almost all* the debt in finite time.

To formalize this idea, fix prices p, q and a portfolio trading plan θ. Define *debt* at date-event s as the value (in units of account) of the obligation to repay on securities held entering date-event s. Because securities are denominated in units of the numeraire commodity, and the price of the numeraire commodity is 1, debt at date-event s is:

$$d_s = -\mathrm{div}_s \theta_{s^-}$$

If this quantity is positive, a trader following the portfolio trading plan θ is in debt entering date-event s. To meet this debt, the trader must raise income from the sale of endowment and/or securities (selling securities is borrowing).

[8]See also Magill and Quinzii (1994).

We constrain debt at date-event s by prescribing a positive upper bound on d_s.[9] (Prescribing a negative upper bound would require traders to save.) We say that the debt $d_s \geq 0$ can be *repaid in finite time from s* if there are consumption and portfolio plans y, φ and a time horizon T such that:

- $p_s \cdot y_s + q_s \cdot \varphi_s + d_s \leq p_s \cdot e^i_s$
- y_σ, φ_σ satisfy the spot budget constraint (1) at every date-event $\sigma \geq s$
- if $\sigma \geq s$ and $t(\sigma) - t(s) \geq T$ then $d_\sigma \leq 0$

That is, the plans meet the liability d_s at the date-event s, meet the spot budget constraints at every date-event following the date-event s and leave no debt at any date-event following s by T or more periods. Define the *finitely effective debt constraints* as:

$$D^i_s = \sup\{d : d \text{ can be repaid in finite time from } s\}$$

Finally, define the *budget set* for trader h at prices p, q as:

$$\begin{aligned} B^i(p,q) \;=\; \Big\{x^i, \theta^i \;:\; & \text{for each } s: \\ & p_s \cdot x^i_s + q_s \cdot \theta^i_s \leq p_s \cdot e^i_s + \mathrm{div}_s \theta^i_{s^-}, \\ & d_\sigma = -\mathrm{div}_\sigma \theta^i_s \leq D^i_\sigma \;\text{ for every } \sigma \in s^+ \Big\} \end{aligned}$$

Note that we constrain trades at date event s by limiting debt at all date events that immediately follow s. (No debt constraint is necessary entering the initial date-event 0 because initial holdings of securities are 0.)

2.7 Equilibrium

An *equilibrium* consists of spot prices p, security prices q, consumption plans (x^i) and portfolio plans (θ^i) such that

- for each s:

$$\sum_i x^i_s = \sum_i e^i_s$$

[9] The reader familiar with Levine and Zame (1996) will note that we use here the opposite sign convention for debt and debt constraints.

- for each s:

$$\sum_i \theta_s^i = 0$$

- for each i:

$$\begin{array}{rcl} (x^i, \theta^i) & \in & B^i(p,q) \qquad \text{and} \\ (y^i, \varphi^i) \in B^i(p,q) & \Rightarrow & U^i(x^i) \geq U^i(y^i) \end{array}$$

That is, commodity markets clear, security markets clear, traders optimize in their budget sets. Levine and Zame (1996) show that (with assumptions weaker than those made here) an equilibrium exists.[10,11]

[10]Existence of equilibrium depends on the assumption that assets are denominated in a numeraire good; without that assumption, only pseudo-equilibria need exist.

[11]As Kubler and Schmedders (1999) show, Markov equilibria need not exist.

3 Idiosyncratic Risk

In this section and the next we address one good economies. We begin by considering economies in which risk is purely idiosyncratic.

Assumption A1 $L = 1$ (one good).

Assumption A2 The social endowment $e = \sum_i e^i_s$ is independent of $s \in S$ (no aggregate risk).

Because the social endowment is constant and the number of traders is finite, endowments must necessarily be correlated across individuals. However, this necessary correlation is purely an artifact of the finiteness of our model. An obvious alternative would be to consider a model with a continuum of traders, in which case independence of individual risks would be consistent with absence of aggregate risk. A result similar to our Theorem A below could be established about such a model. We prefer here the model with a finite number of traders because Levine and Zame (1996) guarantees that an equilibrium exists; to our knowledge, no comparable existence theorems are known for the model with a continuum of traders.[12]

In addition to the previous assumptions about utility functions, we assume the following.

Assumption A3 For each i, Du^i is (weakly) convex.

If utility functions are C^3, Assumption A3 will be satisfied if third derivatives are non-negative (so Assumption A3 is related to a precautionary demand for saving in the sense of Leland (1968)), which will in turn be the case if absolute risk aversion is non-increasing. See Lemma 1 below for the import of Assumption A3 in our context.

Finally, we make one assumption about the asset structure.

[12]Models with a continuum of identical traders were introduced and analyzed by Bewley (1986), but we allow here for heterogeneity across traders.

Assumption A4 At each date-event s, a riskless real bond, yielding one unit of consumption at each date-event $\sigma \in s^+$, is available for trade.

Note that we allow for the possibility that other assets are also traded, and that different assets are traded at different date-events.

We are interested in the nature of equilibrium for discount factors δ close to 1. We therefore fix securities, endowments and period utility functions u^i. For each discount factor $\delta < 1$, write $\mathcal{E}_\delta$ for the economy (with the given securities, endowments and period utility functions) in which traders use the common discount factor δ, and write EQ_δ for the set of equilibria of $\mathcal{E}_\delta$. Because we normalize the spot price of consumption to be identically 1, we henceforth suppress spot prices. An equilibrium $\zeta \in EQ_\delta$ is therefore a triple $\zeta = \left(q_\zeta, (x^i_\zeta), (\theta^i_\zeta)\right)$, of asset prices q_ζ, consumption plans (x^i_ζ) and portfolio plans (θ^i_ζ). Given an equilibrium ζ and a date event $s \in S$, we write $q_{\zeta s}, x^i_{\zeta s}, \theta^i_{\zeta s}$ (rather than $(q_\zeta)_s$ and so forth) for prices, consumption and portfolio choice of the i-th consumer at s. When there is no danger of confusion, we suppress the subscript ζ.

Because individual endowments depend only on the state of the underlying Markov chain, they each have a long run average; write $\overline{e^i}$ for the long run average of e^i.

Assumptions A1 and A2, together with our previous assumptions, imply that, for every δ, the Pareto optimal allocations of $\mathcal{E}_\delta$ coincide with (N-tuples of) fixed shares of the constant social endowment. In particular, for every δ, the perfect risk-sharing allocation $\hat{e} = (\overline{e^1}, \ldots, \overline{e^N})$ at which each trader consumes his long run average endowment, is Pareto optimal.

Our first result asserts that when δ is sufficiently close to 1 (that is, when traders are sufficiently patient), every equilibrium is close to perfect risk sharing, in the sense that (i) equilibrium utilities are close to the utilities of the perfect risk sharing allocation, (ii) the time-discounted probability that equilibrium consumptions deviate from the perfect risk sharing allocation by more than a given amount is small, (iii) the time-discounted probability that equilibrium asset prices deviate from risk neutral pricing by more than a given amount is small.[13]

[13]Because feasible consumptions are bounded by the social endowment, (ii) implies that the time discounted expected deviation of consumptions from perfect risk sharing is small. However, equilibrium asset prices need not be bounded, so (iii) does not imply that the time discounted expected deviation of asset prices from risk neutral pricing is small.

To measure the deviation from perfect risk sharing, fix a discount factor δ and an equilibrium $\zeta = \left(q_\zeta, (x^i_\zeta), (\theta^i_\zeta)\right)$. For $\varepsilon > 0$, write

$$S^c_\varepsilon(\delta;\zeta) = \left\{ s \in S : \exists \text{ trader } i,\ |x^i_{\zeta s} - \bar{e}^i| > \varepsilon \right\}$$

This is the set of date events at which the equilibrium consumption of some trader differs from his long run average consumption by more than ε.

To measure the deviation from risk neutral pricing, fix a discount factor δ and an equilibrium $\zeta = \left(q_\zeta, (x^i_\zeta), (\theta^i_\zeta)\right)$. If s is any date event and φ is a portfolio traded at s, then $E_s(\mathrm{div}_\sigma \varphi)$ is the expected payoff of φ, and $\delta E_s(\mathrm{div}_\sigma \varphi)$ is the discounted expected payoff of φ, which by definition is the risk neutral price. The deviation from risk neutral pricing can therefore be measured by the amount by which the ratio of the equilibrium price to the risk neutral price differs from unity. For $\varepsilon > 0$, write

$$S^p_\varepsilon(\delta;\zeta) = \left\{ s \in S : \exists \text{ portfolio } \varphi,\ \left| \frac{q_{\zeta s} \cdot \varphi}{\delta E_s(\mathrm{div}_\sigma \varphi)} - 1 \right| > \varepsilon \right\}$$

This is the set of date events at which risk neutral pricing of some portfolio fails by more than ε.

Theorem A *If Assumptions A1-A4 are satisfied then:*

(i) *for every trader i:*

$$\lim_{\delta \to 1} \sup_{\zeta \in EQ_\delta} \left| U^i_\delta(x^i_\zeta) - u^i(\bar{e}^i) \right| = 0$$

(ii) *for each* $\varepsilon > 0$*:*

$$\lim_{\delta \to 1} \sup_{\zeta \in EQ_\delta} (1-\delta) \sum_{s \in S^c_\varepsilon(\delta;\zeta)} \delta^{t(s)} \pi_s \ = \ 0$$

(iii) *for each* ε*:*

$$\lim_{\delta \to 1} \sup_{\zeta \in EQ_\delta} (1-\delta) \sum_{s \in S^p_\varepsilon(\delta;\zeta)} \delta^{t(s)} \pi_s \ = \ 0$$

The proof (deferred to Appendix 1) provides a lower bound on equilibrium utility by constructing a budget feasible plan whose utility is almost that of constant average consumption. From this estimate, the nature of the Pareto set allows us to infer (i), and the remaining conclusions follow easily. A crucial step in the argument is establishing that the price of the riskless bond is not much below 1 at every date-event. (Equivalently: the riskless interest rate is not much above 0.) This will be important because the budget feasible plan we construct is financed by borrowing, and a high price (low interest rate) makes borrowing easy.

The price (interest rate) estimate we need is contained in the following lemma, which represents a particular formalization of an intuition common in the finance literature (but not, as far as we can find, established rigorously in any context similar to ours) that a precautionary demand for saving drives down the interest rate. The elegant proof below is a small adaptation of an argument in Hara and Kajii (2000); our original argument was much more cumbersome.

Lemma 1 *Assume A1-A4 hold. Fix a subjective discount factor* δ. *If* $q,(x^i),(\theta^i)$ *is an equilibrium for the economy* $\mathcal{E}_\delta$ *and* q^1 *is the price of the riskless bond, then* $q^1_s \geq \delta$ *at every date-event* $s \in S$. *(That is, the price of the riskless bond is bounded below by the subjective discount factor* δ *at every date-event; equivalently, the riskless interest rate is no greater than the subjective discount rate* $\frac{1}{\delta}-1$ *at every date-event.)*

Proof Fix a date-event $s \in S$, and write q^1_s for the price of the riskless bond at s. For each trader i, an application of the first order conditions for equilibrium together with convexity of period utility functions yields

$$\begin{aligned} q^1_s Du^i(x^i_s) &\geq \delta \sum_{\sigma \in s^+} \pi(\sigma|s) Du^i(x^i_\sigma) \\ &\geq \delta\, Du^i\left(\sum_{\sigma\in s^+}\pi(\sigma|s)x^i_\sigma\right) \end{aligned}$$

Hence

$$q^1_s \geq \delta\, \frac{Du^i\left(\sum_{\sigma\in s^+}\pi(\sigma|s)x^i_\sigma\right)}{Du^i(x^i_s)} \tag{2}$$

The assumption of no aggregate risk entails that

$$\sum_i \sum_{\sigma\in s^+} \pi(\sigma|s)x^i_\sigma = \sum_{\sigma\in s^+}\sum_i \pi(\sigma|s)x^i_\sigma = e_\sigma = e_s = \sum_i x^i_s$$

Hence there is at least one trader j for whom

$$\sum_{\sigma \in s^+} \pi(\sigma|s) x^j_\sigma \leq x^j_s$$

Because period utility functions are concave, Du^j is decreasing, so

$$Du^j \left(\sum_{\sigma \in s^+} \pi(\sigma|s) x^j_\sigma \right) \geq Du^j(x^j_s) \tag{3}$$

Combining the inequalities (2) and (3) we conclude that

$$q^1_s \geq \delta \frac{Du^j \left(\sum_{\sigma \in s^+} \pi(\sigma|s) x^j_\sigma \right)}{Du^j(x^j_s)} \geq \delta$$

which is the desired result. ■

4 Aggregate Risk

As we shall see in Appendix 1, the proof of Theorem A depends on the estimate established in Lemma 1, that the equilibrium riskless interest rate is bounded above by the subjective discount rate, and so is small when traders are patient. When there is aggregate risk, however, this bound need not obtain; when the aggregate endowment is low, many traders will want to borrow, and this demand for loans may drive up the riskless interest rate. A high riskless interest rate interferes with risk sharing because it makes borrowing difficult. The two examples below formalize this intuition, showing that aggregate risk can interfere with the sharing of individual risk, even under rather stringent assumptions on preferences, endowments and the asset structure. In Theorem B, which follows the examples, we show that tradability of aggregate risk is key to almost perfect risk sharing.

Our first two assumptions parallel those of the previous section.

Assumption B1 $L = 1$ (one good).

Assumption B2 For each i, the individual share process $f^i_s = e^i_s / e_s$ is independent of the aggregate endowment process e_s.[14]

We assume that period utility functions are identical across agents and display constant relative risk aversion.[15]

Assumption B3 Utility functions u^i display constant relative risk aversion $\gamma > 0$. That is

$$u^i(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma \neq 1 \\ \\ \log x & \text{if} \quad \gamma = 1 \end{cases}$$

We assume as before that a riskless bond is traded at each date-event.

[14]See the comments in Section 3 about correlation across traders.

[15]When $\gamma \geq 1$, preferences over infinite horizon consumption streams generated by CRRA period utility functions do not satisfy our standing assumptions from Section 2; hence we are not guaranteed that an equilibrium exists in this case.

Assumption B4 At each date-event s, a riskless real bond B yielding one unit of consumption at each date-event $\sigma \in s^+$, is available for trade.

As in Section 3, we are interested in the nature of equilibrium for discount factors δ close to 1. As before, we fix securities, endowments and period utility functions u^i. As before, normalize so that the spot price of consumption is identically 1. For each discount factor $\delta < 1$, write $\mathcal{E}_\delta$ for the economy with the given securities, endowments and period utility functions, in which traders use the common discount factor δ, and write EQ_δ for the set of equilibria of $\mathcal{E}_\delta$. Because endowments are Markov, both endowments and individual endowment shares possess long run averages; write $\overline{e^i}$ for the long run average of e^i and $\overline{f^i}$ for the long run average of f^i.

Assumptions B1-B3, together with our previous assumptions, imply that, for every δ, the set of Pareto optimal allocations of $\mathcal{E}_\delta$ coincides with the set of fixed shares of the *varying* social endowment. In particular, the perfect risk-sharing allocation $\hat{e} = (\overline{f^1}e, \ldots, \overline{f^N}e)$ at which each trader consumes his long run average *share* of the *varying* social endowment, is Pareto optimal (for each δ).

However, even these very strong assumptions are not enough to guarantee that market incompleteness does not matter.

Example 1 We describe an economy satisfying the assumptions B1-B4 above, and in which only the riskless bond is traded, and show that almost perfect risk-sharing does not obtain.

The underlying Markov process has 4 states: the process is i.i.d. with transition probabilities 1/4; the initial state is state 1. There is a single consumption good, and a single asset, a riskless bond. There are 2 traders, with CRRA period utility function $u(x) = x^{1-\gamma}/(1-\gamma), 0 < \gamma < 1$. In states 1, 3 the social endowment is L (low), in states 2, 4 the social endowment is H (high); $H > L > 1$. Trader 1 (henceforward referred to as the large trader) has endowment share $1-\varepsilon$ in states 1, 2 and $1-k\varepsilon$ in states 3, 4; trader 2 (henceforward referred to as the small trader) has endowment share ε in states 1, 2 and $k\varepsilon$ in states 3,4. We take $k > 3$; $\varepsilon > 0$ is a small parameter, to be chosen below. Note that individual endowment shares are independent of the social endowment, as assumed in B2.

As before, we are interested in the behavior of equilibrium for discount factors δ close to 1. We assert that for δ close to 1, *no* equilibrium is close to perfect risk sharing.

The intuition is simple. Imagine first that $\varepsilon = 0$, so that this is a one trader economy. In that case, bond prices are determined by the large trader's marginal utilities at his endowment (which, because $\varepsilon = 0$, coincides with the aggregate endowment). Thus bond prices are

$$q_1^* = q_3^* = \delta\frac{H^{-\gamma}+L^{-\gamma}}{2L^{-\gamma}} < \frac{H^{-\gamma}+L^{-\gamma}}{2L^{-\gamma}}$$
$$q_2^* = q_4^* = \delta\frac{H^{-\gamma}+L^{-\gamma}}{2H^{-\gamma}} > \frac{H^{-\gamma}+L^{-\gamma}}{2H^{-\gamma}}$$

Because $H > L$, if δ is sufficiently close to 1 bond prices satisfy

$$q_1^* = q_3^* < 1 \tag{4}$$
$$q_2^* = q_4^* > 1 \tag{5}$$

Now imagine that $\varepsilon > 0$ but infinitesimal, so that the small trader has no effect on prices. Then the bond price continues to satisfy (4) when the aggregate endowment is L and δ is sufficiently close to 1, whence the riskless interest rate is positive and bounded away from 0 when the social endowment is L. Say the riskless interest rate is at least $\rho > 0$ when the social endowment is L. Because the equilibrium conditions require that the small trader be able to almost repay debt at every date-event, this entails that the debt of the small trader can never be so large that his endowment will not cover the interest on his debt. Thus, the debt of the small trader can never exceed $k\varepsilon L/\rho$, independent of the subjective discount factor δ. On the other hand, consumption smoothing requires the small trader to borrow when his endowment share is small, and in particular, whenever the Markov process is in state 1. Hence, along any history in which the Markov process enters state 1 and then remains in state 1 for a long time, the small trader will be unable to perfectly smooth consumption (because doing so would eventually raise his debt above $k\varepsilon L/\rho$). For δ close to 1, the utility loss from the failure to smooth perfectly along such histories will be non-negligible. Thus, an absolute upper bound on the debt of the small trader implies an absolute upper bound on the ability of the small trader to smooth consumption, and so rules out perfect risk sharing.

Unfortunately, two complications make turning this intuition into rigorous analysis rather difficult. The first complication is that if ε is small but not infinitesimal, equilibrium bond prices may not satisfy (4) when the aggregate endowment is L. Indeed, bond prices may fail this estimate by a great deal at *a few* date-events. If bond prices are occasionally very high (so that interest rates are occasionally very negative), the small trader will be able to repay a very large debt. The second complication is that whether or not equilibrium bond prices satisfy (4) and (5), the interest rate will certainly be positive in some date-events. This leaves open the possibility that the small trader can build a large buffer stock of saving.

Our formal analysis of Example 1 follows the intuition above to establish bounds on the equilibrium debt and saving of the small trader that are *independent of the discount factor* δ. Such bounds imply a limit on equilibrium consumption smoothing of the small trader and hence rule out almost perfect risk sharing. Because the argument is quite involved, we defer the details to Appendix 2. $\diamondsuit$

In treating a framework in which there is aggregate risk, but only riskless bonds are traded, Example 1 parallels Telmer (1993), but reaches the opposite conclusion. Much of the macroeconomics/finance literature (see Lucas (1994) and Heaton and Lucas (1996) for instance), however, treats a framework in both bonds and equity are traded. It seems natural to view the social endowment as the analog of equity in our exchange framework, and thus to make the following assumption.

Assumption B4′ At each date-event s, there are available for trade:

(a) a risky asset A yielding one unit of the social endowment e_σ at each date-event $\sigma \in s^+$

(b) a riskless bond B yielding one unit of consumption at each date-event $\sigma \in s^+$

In Example 1, there are only two aggregate states, so if riskless bonds and the social endowment are both traded then the market for aggregate risk is complete. As Theorem B below demonstrates, this is enough to guarantee that almost perfect risk sharing again obtains. If there are at least 3 aggregate states, however, tradability of riskless bonds and the social endowment is compatible with incompleteness of the market for aggregate risk. Example 2 below suggests that, in such a setting, almost perfect risk sharing again need not obtain. Unfortunately, a rigorous analysis — which would necessarily be much more complicated than the rigorous analysis of Example 1 presented in Appendix 2 — is beyond our present capabilities, so we content ourselves with presenting the (very suggestive) intuition.

Example 2 We describe an economy satisfying the assumptions B1-B3, B4$'$ above, and present the intuition that almost perfect risk-sharing should not obtain.

The underlying Markov process has 6 states. The process is i.i.d.; for $\pi > 0$ to be specified below, $j = 1, 2, 3, 4$ the probability of transiting into state j is $\frac{1-\pi}{4}$, for $j = 5, 6$ the probability of transiting into state j is $\frac{\pi}{2}$; the initial state is state 1. If s is a date-event, write $\omega(s)$ for the state of the Markov process at s. There is a single consumption good. The social endowment is H in states 1, 3, M in states 2, 4 and L in states 5, 6. To simplify computation we take $H = 9, M = 4, L = 1$. There are two traders. Trader 1 (the large trader) has endowment share $1-\varepsilon$ in states 1, 2, 5 and $1-3\varepsilon$ in states 3, 4, 6; trader 2 (the small trader) has endowment share ε in states 1, 2, 5 and 3ε in states 3, 4, 6. The endowment patterns are summarized in the following table.

state	e^1	e^2
1	$9(1-\varepsilon)$	9ε
2	$4(1-\varepsilon)$	4ε
3	$9(1-3\varepsilon)$	$9(3\varepsilon)$
4	$4(1-3\varepsilon)$	$4(3\varepsilon)$
5	$1-\varepsilon$	ε
6	$1-3\varepsilon$	3ε

Traders share the same CRRA period utility function; to simplify computation we take risk aversion $\gamma = 1/2$ so $u(x) = 2x^{1/2}$.

Two assets are available at each date-event s: a risky asset A, yielding the social endowment in each date-event $\sigma \in s^+$, and a riskless bond B yielding one unit of consumption in each date-event $\sigma \in s^+$. The probability π and share ε are parameters, chosen below. Note that individual endowment shares are independent of the social endowment, as assumed in B2.

As before, we are interested in the behavior of equilibrium for discount factors δ close to 1; we suggest that if π, ε are small then no equilibrium is close to perfect risk sharing (independent of δ).

To understand the intuition, suppose as in Example 1 that ε were positive but infinitesimal, so that asset prices are determined by the large trader's marginal utilities at his endowment (which coincides, up to an infinitesimal, with the aggregate endowment). Thus, asset prices, indexed by the aggregate endowment, would be

$$\begin{aligned}
q_H^A &= \delta \cdot \frac{3}{2} \cdot \left[\frac{1-\pi}{2} \cdot \frac{1}{3} + \frac{1-\pi}{2} \cdot \frac{1}{2} + \pi \cdot 1\right] \\
q_M^A &= \delta \cdot 1 \cdot \left[\frac{1-\pi}{2} \cdot \frac{1}{3} + \frac{1-\pi}{2} \cdot \frac{1}{2} + \pi \cdot 1\right] \\
q_L^A &= \delta \cdot \frac{1}{2} \cdot \left[\frac{1-\pi}{2} \cdot \frac{1}{3} + \frac{1-\pi}{2} \cdot \frac{1}{2} + \pi \cdot 1\right] \\
q_H^B &= \delta \cdot \frac{3}{2} \cdot \left[\frac{1-\pi}{2} \cdot 3 + \frac{1-\pi}{2} \cdot 2 + \pi \cdot 1\right] \\
q_M^A &= \delta \cdot 1 \cdot \left[\frac{1-\pi}{2} \cdot 3 + \frac{1-\pi}{2} \cdot 2 + \pi \cdot 1\right] \\
q_L^A &= \delta \cdot \frac{1}{2} \cdot \left[\frac{1-\pi}{2} \cdot 3 + \frac{1-\pi}{2} \cdot 2 + \pi \cdot 1\right]
\end{aligned}$$

Fix a date-event s for which $\omega(s) = 5$, so aggregate endowment is $L = 1$ and the share of the small trader is ε. We assert that, independently of the discount factor δ, it is not possible for the small trader to repay a debt $d_s = 1$ in finite time (and hence not possible for the small trader to repay any larger debt in finite time). As in Example 1, this will rule out perfect risk sharing, because the small trader must borrow when his endowment share is small, and in particular when the Markov process is in state 1. Thus, along histories in which the Markov process enters state 1 and then remains in state 1 for a long time, the small trader cannot perfectly smooth consumption (because doing so would eventually raise his debt above 1). For δ close to 1, the utility loss from failure to smooth perfectly along such histories will be non-negligible. Thus, as in Example 1, an absolute upper bound on the debt of the small trader implies an absolute upper bound on his ability to smooth consumption, and so rules out perfect risk sharing. (Keep in mind that this *assumes* the pricing relationships above and represents an *intuition*, not a rigorous argument.)

To see that it is not possible for the small trader to repay a debt $d_s = 1$ in finite time, independently of δ, suppose there is a plan which repays the debt $d_s = 1$ in finite time. Among all such plans, we consider one which repays fastest; say that repayment is complete (and debt is 0) by date $t(s)+T$ (that is, T dates after s). The crucial property of any such plan is that if $s' \geq s$, $\omega(s') = 5$, and $t(s') - t(s) \leq T$, then debt at s' cannot be less than 1 — else we could shift the plan from s' to begin at s and repay the debt in fewer than T dates.

We reach a contradiction by considering debt at states s' that follow s for which $\omega(s') = 2$ or 5. In particular, let $\tau_5^1 \in s^+$ be the date-event for which $\omega(\tau_5^1) = 5$, and let $\tau_2^1 \in s^+$ be the date-event for which $\omega(\tau_2^1) = 2$. For each $t \geq 1$ let $\tau_2^{t+1} \in {\tau_2^t}^+$ be the date-event for which $\omega(\tau_2^{t+1}) = 2$. Write α_ν, β_ν for the required purchases of the assets A, B at the date-event ν.

First consider the situation at the date-event s. Budget balance requires

$$q_L^A \alpha_s + q_L^B \beta_s + \varepsilon \geq 1$$

Because $\omega(\tau_5^1) = 5$, the crucial property of our plan guarantees that debt is no greater than 1 at τ_5^1, so

$$\alpha_s + \beta_s \leq 1$$

By definition, $\omega(\tau_2^1) = 2$, so acquiring the portfolio $\alpha_s A + \beta_s B$ at s leaves debt $d_{\tau_2^1} = 4\alpha_s + \beta_s$ at τ_2^1.

Now consider the situation at the date-event τ_2^1. Budget balance at τ_2^1 requires that the debt $d_{\tau_2^1} = 4\alpha_s + \beta_s$ be repaid from endowment and further portfolio trades so

$$q_M^A \alpha_{\tau_2^1} + q_M^B \beta_{\tau_2^1} + \varepsilon \geq d_{\tau_2^1}$$

Let $\tau_5^2 \in {\tau_2^1}^+$ be the date-event for which $\omega(\tau_5^2) = 5$. Because $\omega(\tau_5^2) = 5$, the crucial property of our plan guarantees that $d_{\tau_5^2} \leq 1$, so

$$\alpha_{\tau_2^1} + \beta_{\tau_2^1} \leq 1$$

Because $\omega(\tau_2^2) = 2$, acquiring the portfolio $\alpha_{\tau_2^1} A + \beta_{\tau_2^1} B$ at τ_2^1 leaves debt $d_{\tau_2^2} = 4\alpha_{\tau_2^1} + \beta_{\tau_2^1}$ at τ_2^2.

Continuing inductively, we see that $d_{\tau_2^{t+1}} = 4\alpha\tau_2^t + \beta_{\tau_2^t}$ for $t = 1, \ldots, T$.

Now we estimate these various debts. The debt $d_{\tau_2^1}$ is bounded below by the solution $d_2^1(\pi,\varepsilon,\delta)$ to the linear program:

$$\begin{array}{llcl} \text{Choose } \alpha,\beta \text{ to} & & & \\ \text{Minimize} & 4\alpha+\beta & & \\ \text{Subject to} & q_L^A\alpha+q_L^B\beta+\varepsilon & = & 1 \\ & \alpha+\beta & \leq & 1 \end{array} \tag{6}$$

The debt $d_{\tau_2^2}$ is bounded below by the solution $d_2^2(\pi,\varepsilon,\delta)$ to the linear program:

$$\begin{array}{llcl} \text{Choose } \alpha,\beta \text{ to} & & & \\ \text{Minimize} & 4\alpha+\beta & & \\ \text{Subject to} & q_M^A\alpha+q_M^B\beta+\varepsilon & \geq & d_2^1(\pi,\varepsilon,\delta) \\ & \alpha+\beta & \leq & 1 \end{array} \tag{7}$$

And by induction, for each $t \leq T-1$, the debt $d_{\tau_2^{t+1}}$ is bounded below by the solution $d_2^{t+1}(\pi,\varepsilon,\delta)$ to the linear program:

$$\begin{array}{llcl} \text{Choose } \alpha,\beta \text{ to} & & & \\ \text{Minimize} & 4\alpha+\beta & & \\ \text{Subject to} & q_M^A\alpha+q_M^B\beta+\varepsilon & \geq & d_2^t(\pi,\varepsilon,\delta) \\ & \alpha+\beta & \leq & 1 \end{array} \tag{8}$$

Simple but messy calculations show that

$$d_2^1(0,0,1) = \frac{46}{25}\ ,\ d_2^2(0,0,1) = \frac{2046}{625}$$

The solutions to the linear programs (6)-(8) are continuous functions of the parameters π,ε,δ. Since $\frac{2046}{625} > \frac{46}{25}$, it follows that for π,ε sufficiently small and δ sufficiently close to 1 we have

$$d_2^1(\pi,\varepsilon,\delta) > \frac{45}{25}\ ,\ d_2^2(\pi,\varepsilon,\delta) \geq d^1(\pi,\varepsilon,\delta)$$

Our inductive construction then guarantees that

$$d_2^T(\pi,\varepsilon,\delta) \geq d_2^{T-1}(\pi,\varepsilon,\delta) \geq \ldots \geq d_2^1(\pi,\varepsilon,\delta) > \frac{45}{25}$$

Since $d_{\tau_2^T} \geq d_2^T(\pi,\varepsilon,\delta)$, it follows in particular that $d_{\tau_2^T} > 0$. But this contradicts our assumption that the plan repays debt in T dates from s. We conclude that a debt $d_s = 1$ or greater cannot be repaid in finite time from date-event s.

As in Example 1, this suggests that formal analysis will show that almost perfect risk sharing cannot be achieved. ♢

As these examples suggest, when there is aggregate risk, almost perfect risk sharing requires that the "right" securities (or portfolios) be traded. Which securities are the "right" ones will depend on utility functions, but we can guarantee that the "right" securities are traded if we require that *all* derivatives on the social endowment are traded, so that the market for aggregate risk is complete.

Write Υ for the set of possible values of the social endowment e. Because endowments are Markov, Υ is a finite set.[16] The following assumption is that all derivatives on the social endowment are traded at every date event.[17] As Ross (1976) shows, this is implied by the assumption that options on the social endowment are traded at every date-event.

Assumption B4″ For each date-event s and for each $\upsilon \in \Upsilon$, there is a portfolio Γ_s^υ of securities such that

$$\mathrm{div}_\sigma(\Gamma_s^\upsilon) = \begin{cases} 1 & \text{if } e_s = \upsilon \\ 0 & \text{if } e_s \neq \upsilon \end{cases}$$

Theorem B, paralleling Theorem A of the previous section, asserts that when traders are sufficiently patient every equilibrium is close to perfect risk sharing, in the sense that (i) equilibrium utilities are close to the utilities of the perfect risk sharing allocation, (ii) the time-discounted probability that equilibrium consumptions deviate from the perfect risk sharing allocation by more than a given amount is small, (iii) the time-discounted probability that equilibrium asset prices deviate from marginal rates of substitution at the perfect risk sharing allocation by more than a given amount is small. (In the absence of aggregate risk, this reduces to risk neutral pricing, as in Theorem A.) As before, fix a discount factor δ and an

[16] If Υ and s^+ have the same cardinality, availability of all derivatives on the social endowment is equivalent to completeness of intertemporal markets. In the typical case, however, Υ is much smaller than s^+.

[17] In fact, the proof requires only that certain very particular derivatives are traded.

equilibrium $\zeta = \left(q_\zeta, (x^i_\zeta), (\theta^i_\zeta)\right)$. As before, for each $\varepsilon > 0$ we define:

$$S^{c}_{\varepsilon}(\delta;\zeta) = \left\{ s \in S : \exists \text{ trader } i, |x^i_{\zeta s} - \overline{f^i} e_s| > \varepsilon \right\}$$

$$S^{p}_{\varepsilon}(\delta;\zeta) = \left\{ s \in S : \exists \text{ portfolio } \varphi, \left| \frac{q_{\zeta s} \cdot \varphi}{\delta E_s \left[\left(\frac{e_s}{e_\sigma} \right)^{\gamma} \mathrm{div}_\sigma \varphi \right]} - 1 \right| > \varepsilon \right\}$$

The first set is the set of date events at which some consumer's equilibrium consumption differs from his perfect risk sharing consumption by more than ε; the second set is the set of date events at which risk neutral pricing of some portfolio fails by more than ε.

Theorem B *If Assumptions B1-B3 and B4$''$ are satisfied, then:*

(i) *for every trader i,*

$$\lim_{\delta \to 1} \sup_{\zeta \in EQ_\delta} \left| U^i_\delta(x^i) - U^i_\delta(\overline{f^i} e_s) \right| = 0$$

(ii) *for each* $\varepsilon > 0$*:*

$$\lim_{\delta \to 1} \sup_{\zeta \in EQ_\delta} (1-\delta) \sum_{s \in S^c_\varepsilon(\delta;\zeta)} \delta^{t(s)} \pi_s = 0$$

(iii) *for each* $\varepsilon > 0$*:*

$$\lim_{\delta \to 1} \sup_{\zeta \in EQ_\delta} (1-\delta) \sum_{s \in S^p_\varepsilon(\delta;\zeta)} \delta^{t(s)} \pi_s = 0$$

As with Theorem A, the proof rests on a price estimate. Recall that Υ is the range of the social endowment process. For $s \in S, \upsilon \in \Upsilon$, write $s^+(\upsilon)$ for the set of date-events $\sigma \in s^+$ at which $e_\sigma = \upsilon$, and write $\pi(\upsilon|s)$ for the conditional probability that the social endowment is υ following the date-event s. Assumption B4$''$ guarantees that a portfolio Γ^υ_s promising one unit of consumption in date-events $\sigma \in s^+(\upsilon)$ and 0 otherwise is traded; the following result (which relies on the assumption of CRRA utilities) provides a bound on its equilibrium price. The proof parallels closely the proof of Lemma 1.

Lemma 2 *Assume B1-B3 and B4$''$ hold. Fix a subjective discount factor δ. If $q, (x^i), (\theta^i)$ is an equilibrium for the economy $\mathcal{E}_\delta$, then*

$$q_s \cdot \Gamma_s^\upsilon \geq \delta\pi(\upsilon|s)\left(\frac{\upsilon}{e_s}\right)^{-\gamma}$$

at every date-event $s \in S$.

Proof Differentiation shows that if u is a CRRA utility function with risk aversion γ then $D^3u(x) = \gamma^2 x^{-\gamma-2}$, which is strictly positive. In particular, Du is convex, so we can follow the same plan as in the proof of Lemma 1. Fix a date-event $s \in S$ and a consumption level $\upsilon \in \Upsilon$. For each trader i, an application of the first order conditions for equilibrium, together with convexity of the period utility function yields

$$\begin{aligned}
(q_s \cdot \Gamma_s^\upsilon)Du^i(x_s^i) &\geq \delta \sum_{\sigma \in s^+(\upsilon)} \pi(\sigma|s)Du^i(x_\sigma^i) \\
&\geq \delta\pi(\upsilon|s) \sum_{\sigma \in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)} Du^i(x_\sigma^i) \\
&\geq \delta\pi(\upsilon|s) Du^i\left(\sum_{\sigma \in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)} x_\sigma^i\right)
\end{aligned}$$

We can write trader i's consumption as shares of the social consumption: $x_s^i = \kappa_s^i e_s$, $x_\sigma^i = \kappa_\sigma^i e_\sigma = \kappa_\sigma^i \upsilon$. Substituting gives:

$$(q_s \cdot \Gamma_s^\upsilon)Du^i(\kappa_s^i e_s) \geq \delta\pi(\upsilon|s) Du^i\left(\sum_{\sigma \in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)} \kappa_\sigma^i \upsilon\right)$$

Equivalently

$$q_s \cdot \Gamma_s^\upsilon \geq \delta\pi(\upsilon|s) \frac{Du^i\left(\sum_{\sigma \in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)} \kappa_\sigma^i \upsilon\right)}{Du^i(\kappa_s^i e_s)} \tag{9}$$

Note that

$$\begin{aligned}\sum_i \kappa_s^i &= 1 \\ \sum_i \kappa_\sigma^i &= 1 \text{ for each } \sigma \\ \sum_{\sigma\in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)} &= 1\end{aligned}$$

Hence

$$\sum_i \sum_{\sigma\in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)}\kappa_\sigma^i = \sum_{\sigma\in s^+(\upsilon)} \sum_i \frac{\pi(\sigma|s)}{\pi(\upsilon|s)}\kappa_\sigma^i = 1 = \sum_i \kappa_s^i$$

It follows that there is at least one trader j for whom

$$\sum_{\sigma\in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)}\kappa_\sigma^j \leq \kappa_s^j$$

Because u^j is concave, Du^j is decreasing so

$$Du^j\left(\sum_{\sigma\in s^+(\upsilon)} \frac{\pi(\sigma|s)}{\pi(\upsilon|s)}\kappa_\sigma^j\upsilon\right) \geq Du^j(\kappa_s^j\upsilon) \tag{10}$$

Combining the inequalities (9) and (10) we find that

$$q_s\cdot\Gamma_s^\upsilon \geq \delta\pi(\upsilon|s)\frac{Du^j(\kappa_s^j\upsilon)}{Du^j(\kappa_s^j e_s)}$$

Because $Du^j = x^{-\gamma}$, we obtain

$$\begin{aligned}q_s\cdot\Gamma_s^\upsilon &\geq \delta\pi(\upsilon|s)\frac{Du^j(\kappa_s^j\upsilon)}{Du^j(\kappa_s^j e_s)} \\ &= \delta\pi(\upsilon|s)\frac{(\kappa_s^j\upsilon)^{-\gamma}}{(\kappa_s^j e_s)^{-\gamma}} \\ &= \delta\pi(\upsilon|s)\frac{\upsilon^{-\gamma}}{e_s^{-\gamma}} \\ &= \delta\pi(\upsilon|s)\left(\frac{\upsilon}{e_s}\right)^{-\gamma}\end{aligned}$$

which is the desired result. ■

We have assumed here that individual shares are independent of the social endowment. The polar opposite assumption would be that individual shares are perfectly correlated with the social endowment. In that case, there would be no idiosyncratic risk, and derivatives on the social endowment would provide perfect risk sharing at every date-event. In particular, the conclusions of Theorem B would obtain in this case too. We conjecture that the conclusions of Theorem B obtain in the intermediate cases also, without any assumption of independence or correlation.

5 Two Goods

We present here a simple example to demonstrate that when there are two consumption goods, market incompleteness may matter a great deal even if there is no aggregate risk — indeed, even if there is no (fundamental) risk at all.

Example 3 Fix any static 2 trader/2 commodity economy with the following properties:

- utility functions u^1, u^2 are smooth and strictly concave
- endowments w^1, w^2 are strictly positive and $w^1 + w^2 = (1,1)$
- there are three Walrasian equilibria:
  p_D, (x^1_D, y^1_D), (x^2_D, y^2_D); p_M, (x^1_M, y^1_M), $(x^2_M y^2_M)$; p_U, (x^1_U, y^1_U), (x^2_U, y^2_U)
- equilibrium consumptions are strictly positive and $x^1_D < x^1_M < x^1_U$
- equilibrium marginal utilities satisfy:

$$\begin{aligned}
\frac{\partial u^1}{\partial x^1}(x^1_M, y^1_M) &= \frac{1}{2}\left[\frac{\partial u^1}{\partial x^1}(x^1_D, y^1_D) + \frac{\partial u^1}{\partial x^1}(x^1_U, y^1_U)\right] \\
\frac{\partial u^2}{\partial x^2}(x^2_M, y^2_M) &= \frac{1}{2}\left[\frac{\partial u^2}{\partial x^2}(x^2_D, y^2_D) + \frac{\partial u^2}{\partial x^2}(x^2_U, y^2_U)\right]
\end{aligned}$$

The underlying Markov chain has two states $\{U, D\}$; transition probabilities are identically 1/2 (so the process is i.i.d.). The information tree S for the dynamic economy therefore has 2 branches at every node. If $s \in S$ is any node, we can write $s^+ = \{s_U, s_D\}$ where $\omega(s_U) = U, \omega(s_D) = D$ (i.e., the underlying Markov process is in state U at the node s_U and in state D at the node D); the conditional probabilities are $\pi(s_U|s) = \pi(s_D|s) = 1/2$, independent of s. In this notation, $0^+ = \{0_U, 0_D\}$ is the set of nodes that follow the initial node 0. Write $\mathcal{U}$ for the set of nodes that follow 0_U and $\mathcal{D}$ for the set of nodes that follow 0_D.

There are two consumption goods. For $i = 1, 2$, endowments are $w^i_s = w^i$ and, when the subjective discount factor is δ. Utility functions are:

$$U^i(x,y) = (1-\delta)\sum_{t=0}^{\infty}\delta^t \sum_{t(s)=t}\left(\frac{1}{2}\right)^t u^i(x_s, y_s)$$

That is, we simply replicate the static economy at each date-event; there is no fundamental uncertainty — but the underlying Markov chain provides a source of *sunspots*.

Fix a discount factor $\delta < 1$. The dynamic economy has many equilibria, including sunspot equilibria in which the underlying Markov chain serves as a coordination device. One such equilibrum may be described as follows:

- at the initial date-event 0, spot prices are p_M, consumption of trader i is (x^i_M, y^i_M), the bond price is equal to the subjective discount factor δ, the bond is not traded;
- at date-events $s \in \mathcal{U}$, spot prices are p^U, consumption of trader i is (x^i_U, y^i_U), the bond price is equal to the subjective discount factor δ, the bond is not traded;
- at date-events $s \in \mathcal{D}$, spot prices are p^D, consumption of trader i is (x^i_D, y^i_D), the bond price is equal to the subjective discount factor δ, the bond is not traded.

We leave to the reader the simple task of using the properties of the static economy to verify that this is an equilibrium of the infinite horizon economy.

For this equilibrium, consumptions in every date-event are Pareto optimal in the static economy, but consumptions at date-events in $\mathcal{U}$ differ from consumptions at date-events in $\mathcal{D}$, Because utility functions are strictly concave, equilibrium consumptions are certainly not Pareto optimal. Because equilibrium consumptions are independent of δ, equilibrium utilities certainly *do not* approach Pareto optimal utilities as $\delta \to 1$.

6 Conclusion

We have argued here that, in a one-good economy populated by infinitely-lived, patient traders, market incompleteness will not matter if shocks are transient and risk is purely idiosyncratic. However, market incompleteness will matter if there is aggregate risk and the "wrong" assets are traded, or if there is more than one good. Aggregate risk matters because it affects asset prices. Multiple consumption goods matter because commodity prices provide another source of untraded risk. As Example 3 demonstrates clearly, the absence of some financial markets weakens the connection between spot prices at various date-events, and therefore expands the role of expectations — even though expectations are correct in equilibrium. The work of Farmer (1997) on multiple equilibria gives a different perspective.

Perhaps the most important implication of our work is that the effects of market structure may be much different in a dynamic setting than in a static (or short horizon) setting. In particular, the incentives for financial innovation may be quite different when dynamic behavior is taken into account.

Two limitations of our work are worth noting. The first is that we treat only exchange economies with perishable goods; interesting extensions would incorporate production, durable goods, human capital and growth. The second is that we do not provide numerical estimates or rates of convergence. In particular, we do not estimate the utility consequences of market incompleteness for given subjective discount factors. Our methods could certainly be adapted to provide such estimates, but the estimates obtained would not be very good. We suspect that sharper methods — and perhaps more stringent assumptions — will be necessary to provide truly useful estimates. For some estimates in our framework, see Kubler and Schmedders (2000); for some estimates in a rather different framework, see Willen (1998).

Appendix 1: Proofs

An overview of the proofs may help guide the reader. To prove Theorem A, we fix a discount factor and an equilibrium. For each trader, we construct an alternative plan of consumption and portfolio trades. Before a specified stopping time, defined in terms of a debt limit, this alternative plan calls for consuming slightly less than the perfect risk sharing quantity, borrowing when necessary and repaying the debt when possible; after the stopping time, it calls for consumption slighly less than the endowment and repaying the debt. A probabilistic estimate (Lemma 4) shows that, if the discount factor is sufficiently close to 1, it is very likely that the stopping time is not reached for many periods. If follows, therefore, that the alternative plan yields almost the utility of the perfect risk sharing allocation. Because equilibrium plans are optimal, the equilibrium plan must yield at least this much utility. Because this conclusion obtains for every trader, the nature of the Pareto set (implied by strict concavity of utility functions) guarantees that no trader can obtain utility much greater than perfect risk sharing. Combining these two inequalities yields (i). The first order conditions then yield (ii), (iii).

Because the alternative plan described above is financed by trading the riskless bond, this argument requires an estimate for the price of the riskless bond. Lemma 1 provides the estimate we need: at every date-event, the price of the riskless bond is not much below 1 (equivalently, the riskless interest rate is not below 0). This one-sided estimate is good enough, because the alternative plan requires borrowing but not saving; a high price (low interest rate), makes borrowing easy. As Example 1 demonstrates, this lower bound for the price depends on the absence of aggregate risk. When there is aggregate risk, the interest rate depends on the realization of the social endowment: when the social endowment is low many traders want to borrow and the demand for loans drives up the interest rate. A high interest rate makes it difficult to borrow, and hence to finance a consumption plan that is close to the perfect risk sharing consumption plan. However, with the assumptions of Theorem B, such a consumption plan can be financed by trading in a particular derivative of the social endowment, carefully chosen to match marginal utilities and prices; the price estimate for this derivative follows from Lemma 2.

Because the proofs of Theorems A and B are so similar, we have arranged the following discussion to avoid redundancy. Our first task is to establish a probabilistic estimate; for this we need a version of thecentral limit theorem for functions of a finite Markov chain. Lemma 3 below, which is Theorem (3), p.83 in

Freedman (1983), is just what we need.[18,19] Following common mathematical usage, we write $[x]$ for the greatest integer not exceeding x.

Lemma 3 *Consider a recurrent Markov chain with finite space Ω. Let $F : \Omega \to \mathbf{R}$ be a real-valued function on Ω whose long-run average is 0. Let Ω^∞ be the space of all infinite sequences of elements of Ω. Fix a state $\xi \in \Omega$ and let π be its stationary probability. For each $\sigma \in \Omega^\infty$ and each integer t, let $k(t), \ell(t)$ be (respectively) the t-th and $t+1$-st occurrences of the state ξ in the sequence σ; define the random variable Y_t on Ω^∞ by*

$$Y_t(\sigma) = \sum_{k(t) \le n < \ell(t)} F(\sigma_n)$$

(If ξ does not occur at least t times in the sequence σ, then the sum defining Y_t is empty, so $Y_t = 0$.) Then

$$T^{*-\frac{1}{2}} \max \left\{ \left| \sum_{j=0}^{T} F(\sigma_j) - \sum_{i=t}^{[T\pi]} Y_t : 0 \le T \le T^* \right| \right\} \to 0 \textit{ in probability}$$

as $T^* \to \infty$.

The following lemma provides the probabilistic estimate we require.

Lemma 4 *Fix a discount factor $\delta < 1$, real numbers $C, g > 0$, and a trader i. Let $\overline{f^i}$ be the long run average of the share process $f_s^i = e_s^i / e_s$. Define*

$$z_s = \frac{1}{\delta} \left(\overline{f^i} - \frac{e_s^i}{e_s} \right) e_s^g \; , \; Z^T(h) = \sum_{t=1}^{T} z_{h_t}$$

Let $T^*(\delta)$ be the greatest integer not exceeding $(1-\delta)^{-3/2}$ and let

$$\mathcal{H}^*(\delta) = \left\{ h \in \mathcal{H} : \left| Z^T(h) \right| < \frac{C}{1-\delta} \textit{ for } 1 \le T < T^*(\delta) \right\}$$

Then

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[18]Freedman, following an idea of Chung (1960), allows for a Markov chain with a countably infinite state space. In that more general context, he imposes expectation and variance conditions that are automatically satisfied when the state space is finite.

[19]We thank Andrew Postlewaite and an anonymous referee for directing us to Freedman.

(i) $\lim_{\delta\to 1} \delta^{T^*(\delta)} = 0$

(ii) $\lim_{\delta\to 1} \mathrm{Prob}[\mathcal{H}^*(\delta)] = 1$

Proof Part (i) follows immediately from L'Hospital's rule and the definition of $T^*(\delta)$.

To see part (ii), note that by definition $T^*(\delta) \le 1 + (1-\delta)^{-3/2}$, so $\mathcal{H}^*(\delta)$ contains the set of histories for which

$$\max\left\{ \frac{\left|Z^T(h)\right|}{T^*(\delta)-1} : 1 \le T \le T^*(\delta) \right\} < \frac{C}{(T^*(\delta)-1)^{1/3}}$$

For $\omega \in \Omega$, let $s \in S$ be any node for which the corresponding state is ω and define $F(\omega) = z_s$. Because the endowment process is Markov and the long-run average of z_s is 0, Lemma 3 entails that

$$T^*(\delta)^{-1/2} \max\left\{ \left| Z^T - \sum_{t=1}^{[T\pi]} Y_t \right| : 1 \le T \le T^*(\delta) \right\} \to 0 \text{ in probability}$$

By construction, the random variables Y_t are i.i.d.; keeping in mind that $T^*(\delta) \to \infty$ as $\delta \to 1$, the rate of convergence in the usual strong law of large numbers (or central limit theorem) yields

$$T^*(\delta)^{-1/2} \max\left\{ \left| \sum_{t=1}^{[T\pi]} Y_t \right| : 1 \le T \le T^*(\delta) \right\} \to 0 \text{ in probability}$$

as $\delta \to 1$. The desired result (ii) now follows from the triangle inequality. ■

We are now ready to begin the proof of Theorems A and B. In order to give a unified argument, we abstract the common elements of the two settings. We take as given a parameter g and a family $\{\Delta_s : s \in S\}$ of portfolios satisfying two properties:

(i) for every $s \in S$ and $\sigma \in s^+$,

$$\mathrm{div}_\sigma \Delta_s = \left(\frac{e_\sigma}{e_s}\right)^{1-g}$$

(ii) for every equilibrium of $\mathcal{E}_\delta$ and every $s \in S$

$$q_s \cdot \Delta_s \geq \delta$$

To obtain Theorem A from the argument given, take $g = 1$ and for each s take Δ_s to consist of one unit of the riskless bond. To obtain Theorem B from the argument given, take $g = 1 - \gamma$ and for each s take

$$\Delta_s = \sum_{\upsilon \in \Upsilon} \left(\frac{e_\upsilon}{e_s} \right)^{1-g} \Gamma_s^\upsilon$$

In either case, property (i) follows immediately and property (ii) follows from Lemma 1 or 2, as appropriate.

Proof of Theorems A and B Fix an equilibrium. We provide lower bounds on equilibrium utilities by constructing alternative plans which are feasible and approximate perfect risk sharing.

Write

$$m = \min_{i,s} e_s^i \ , M = \max_s e_s$$

Fix a trader i and a real number $\eta > 0$. Choose $\bar{\varepsilon} > 0$ so small that

$$\begin{aligned} m - \bar{\varepsilon} M^{1-g} &> 0 \\ u^i(m - \bar{\varepsilon} M^{1-g}) &> u^i(m) - \frac{\eta}{4} \end{aligned}$$

Because u^i is concave, the latter inequality implies that

$$u^i(c - c') > u^i(c) - \frac{\eta}{4} \tag{11}$$

whenever $c \geq m$ and $c' \leq \bar{\varepsilon} M^{1-g}$.

In what follows, we are interested in the limit as $\delta \to 1$, so there is no loss in assuming throughout that

$$\frac{\bar{\varepsilon}}{2(1-\delta)} m^{1-g} > M - m \tag{12}$$

The alternative plan y, φ calls for consumption of a target quantity until a certain stopping time, which occurs when a given debt limit has been reached:

the *target consumption* at $s \in S$ is $c_s = \bar{f}^i e_s - \bar{\varepsilon} e_s^{1-g}$
the *debt limit* is $d^* = \frac{\bar{\varepsilon}}{2(1-\delta)} m^{1-g}$
the *stopping time* is the first date-event at which $d_s \geq d^*$

Define y_s, φ_s in the following way:

- If $d_\sigma < d^*$ for every $\sigma \leq s$ and $e_s^i / e_s < \bar{f}^i$, set

$$\begin{aligned} y_s &= c_s \\ \varphi_s &= -\frac{1}{q_s \cdot \Delta_s}(d_s + c_s - e_s^i)\Delta_s \end{aligned}$$

  That is, if debt has always been below the debt limit and the endowment share is less than the long run average, consume the target consumption, borrowing to do so.

- If $d_\sigma < d^*$ for every $\sigma \leq s$ and $e_s^i / e_s \geq \bar{f}^i$, set

$$\begin{aligned} y_s &= c_s \\ \varphi_s &= -\frac{1}{q_s \cdot \Delta_s}(d_s + c_s - e_s^i)^+ \Delta_s \end{aligned}$$

  That is, if debt has always been below the debt limit and the endowment share is equal to or greater than the long run average, consume the target consumption, repay some of the debt and roll over the remainder — but do not save.[20]

- If $d_\sigma \geq d^*$ for some $\sigma \leq s$ set

$$\begin{aligned} y_s &= e_s^i - \bar{\varepsilon} e_s^{1-g} \\ \varphi_s &= -\frac{1}{q_s \cdot \Delta_s}(d_s - \bar{\varepsilon} e_s^{1-g})^+ \Delta_s \end{aligned}$$

  That is, if debt has ever reached the debt limit, consume slightly less than the endowment, using the difference to pay off some of the debt, and roll over the remaining debt — but do not save.[20]

[20] Of course it is not optimal for trader *i* not to save when possible — but y^i, φ^i is not intended to be an optimal plan, only an alternative we use to provide a lower bound on equilibrium utility. We can estimate accumulation of debt because we have a lower bound on asset prices, but we cannot estimate accumulation of saving because we do not have an upper bound on asset prices. We find it simpler, therefore, to avoid saving entirely.

We will show below that y, φ is a feasible plan and that it yields consumption c_s at "most" date events. If $\bar{\varepsilon}$ is small, then c_s is only slightly less than $\overline{f^i} e_s$, and $u^i(c_s)$ is only slightly less than $u^i(\overline{f^i} e_s)$, so y, φ yields utility almost as large as the perfect risk sharing consumption.

Two preliminary calculations will be useful.

1) Fix $\sigma \in S, \tau \in \sigma^+$. Suppose trader i holds debt d_σ entering σ, consumes c, and trades only the portfolio Δ_σ. If $d_\sigma + c - e^i_\sigma < 0$ then all debt is repaid and debt at succeeding date-events is 0. If $d_\sigma + c - e^i_\sigma \geq 0$ then the spot budget constraint at σ entails that trader i must sell $[1/(q_\sigma \cdot \Delta_\sigma)][d_\sigma + c - e^i_\sigma]$ units of the portfolio Δ_σ. Thus debt at $\tau \in \sigma^+$ is given by the *evolution equation*:

$$d_\tau = \frac{1}{q_\sigma \cdot \Delta_\sigma}(d_\sigma + c - e^i_\sigma)^+ \left(\frac{e_\tau}{e_\sigma}\right)^{1-g} \tag{13}$$

2) Consider any date-event σ at which $d_\sigma < [\bar{\varepsilon}/(1-\delta)] e_\sigma^{1-g}$. Suppose trader i repays $\bar{\varepsilon} e_\sigma^{1-g}$ of the debt at σ and consumes $e^i_\sigma - \bar{\varepsilon} e_\sigma^{1-g}$. Write

$$\beta = \frac{\bar{\varepsilon}}{1-\delta} e_\sigma^{1-g} - d_\sigma$$

Applying the evolution equation (13) and recalling that $q_\sigma \cdot \Delta_\sigma \geq \delta$ we find that for every $\tau \in \sigma^+$:

$$\begin{aligned} d_\tau &= \frac{1}{q_\sigma \cdot \Delta_\sigma}(d_\sigma - \bar{\varepsilon} e_\sigma^{1-g}) \left(\frac{e_\tau}{e_\sigma}\right)^{1-g} \\ &\leq \frac{1}{\delta}\left(\frac{\bar{\varepsilon}}{1-\delta} e_\sigma^{1-g} - \beta - \bar{\varepsilon} e_\sigma^{1-g}\right)\left(\frac{e_\tau}{e_\sigma}\right)^{1-g} \\ &\leq \frac{\bar{\varepsilon}}{1-\delta} e_\tau^{1-g} - \frac{1}{\delta}\left(\frac{e_\tau}{e_\sigma}\right)^{1-g} \beta \end{aligned}$$

With these calculations in hand, we show that y, φ is a feasible plan. Our construction guarantees that y, φ satisfies the spot budget constraints. To show that satisfies the debt constraints, fix a date-event s. If $d_{s'} < d^*$ for every $s' \leq s$, then in particular $d_s < d^* \leq [\bar{\varepsilon}/(1-\delta)] e_s^{1-g}$. Calculation 2) above shows that repaying $\bar{\varepsilon} e_\sigma^{1-g}$ and consuming $e^i_\sigma - \bar{\varepsilon} e_\sigma^{1-g}$ at every $\sigma \geq s$ will repay this debt in finite time. On the other hand, if $d_{s'} \geq d^*$ for some $s' \leq s$, let s'' be the first such date-event, and let s''^{-} be the date-event immediately preceding s''. By assumption, $d_{s''^{-}} < d^*$ so the evolution equation, together with the inequality (12), shows that $d_{s''} < [\bar{\varepsilon}/(1-\delta)]e_s^{1-g}$. The plan y, φ calls for repayment to begin at s'' and continue forever, so calculation 2) above shows that the specified plan d_s repays the debt d_s in finite time. We conclude that y, φ is a feasible plan.

Set $C = \frac{1}{4}\bar{\varepsilon}m^{1-g}$. As in Lemma 4, let $T^*(\delta)$ be the greatest integer less than $(1-\delta)^{-3/2}+1$. Let $\mathcal{H}_1 \subset \mathcal{H}$ be the set of histories for which the stopping time does not occur before $t = T^*(\delta)$ (i.e., the set of histories for which debt does not exeed d^* before time $t = T^*(\delta)$). To estimate $U_\delta^i(y)$, we estimate utility conditional on good histories (those belonging to $\mathcal{H}_1$) and then conditional on bad histories (those not belonging to $\mathcal{H}_1$), and estimate the probabilities that the history is good or bad.

For $h \in \mathcal{H}_1$, consumption is $\overline{f^i}e_s - \bar{\varepsilon}e_s^{1-g}$ when $t < T^*(\delta)$ and at least $m - \bar{\varepsilon}M^{1-g}$ when $t \geq T^*(\delta)$. The utility $U_\delta^i(\text{good})$ conditional on such a good history h is at least

$$\begin{aligned}
U_\delta^i(\text{good}) \;\geq\; & (1-\delta)\sum_{t=0}^{T^*(\delta)-1}\delta^t u^i(\overline{f^i}e_{h_t} - \bar{\varepsilon}e_{h_t}^{1-g}) \\
& + (1-\delta)\sum_{t=T^*(\delta)}^{\infty}\delta^t u^i(m-\bar{\varepsilon}M^{1-g}) \\
\geq\; & (1-\delta)\sum_{t=0}^{\infty}\delta^t u^i(\overline{f^i}e_{h_t} - \bar{\varepsilon}e_{h_t}^{1-g}) \\
& - (1-\delta)\sum_{t=T^*(\delta)}^{\infty}\delta^t u^i(M) \\
& + (1-\delta)\sum_{t=T^*(\delta)}^{\infty}\delta^t u^i(m-\bar{\varepsilon}M^{1-g}) \\
\geq\; & (1-\delta)\sum_{t=0}^{\infty}\delta^t u^i(\overline{f^i}e_{h_t} - \bar{\varepsilon}e_{h_t}^{1-g}) \\
& - \delta^{T^*(\delta)}[u^i(M) - u^i(m-\bar{\varepsilon}M^{1-g})] \\
=\; & U_\delta^i(\overline{f^i}e - \bar{\varepsilon}e^{1-g}) \;-\; \delta^{T^*(\delta)}[u^i(M) - u^i(m-\bar{\varepsilon}M^{1-g})]
\end{aligned}$$

For $h \notin \mathcal{H}_1$, consumption is at least $m - \bar{\varepsilon}M^{1-g}$ at every date-event. Hence the utility $U^i_\delta(\text{bad})$ conditional on such a bad history is at least

$$U^i_\delta(\text{bad}) \geq (1-\delta)\sum_{t=0}^{\infty} \delta^t u^i(m - \bar{\varepsilon} M^{1-g}) = u^i(m - \bar{\varepsilon} M^{1-g})$$

Combining these two estimates (and keeping in mind that utility could be negative), we conclude that

$$\begin{aligned}
U^i_\delta(y) &\geq \text{Prob}(\mathcal{H}_1) U^i_\delta(\text{good}) \\
&\quad + [1 - \text{Prob}(\mathcal{H}_1)] U^i_\delta(\text{bad}) \\
&\geq \text{Prob}(\mathcal{H}_1) \left\{ U^i_\delta(\bar{f}^i e - \bar{\varepsilon} e^{1-g}) - \delta^{T^*(\delta)} [u^i(M) - u^i(m - \bar{\varepsilon} M^{1-g})] \right\} \\
&\quad + [1 - \text{Prob}(\mathcal{H}_1)] u^i(m - \bar{\varepsilon} M^{1-g}) \\
&\geq \text{Prob}(\mathcal{H}_1) U^i_\delta(\bar{f}^i e - \bar{\varepsilon} e^{1-g}) \\
&\quad - \delta^{T^*(\delta)} [u^i(M) - u^i(m - \bar{\varepsilon} M^{1-g})] \\
&\quad - [1 - \text{Prob}(\mathcal{H}_1)] \, |u^i(m - \bar{\varepsilon} M^{1-g})| \qquad (14)
\end{aligned}$$

To estimate $\text{Prob}(\mathcal{H}_1)$, define $\mathcal{H}(\delta)$ be as in Lemma 4. We claim that $\mathcal{H}(\delta) \subset \mathcal{H}_1$. To see this, fix a history $h \in \mathcal{H}(\delta)$ and a date $T \leq T^*(\delta)$. For every date-event s, our assumptions guarantee that $q_s \cdot \Delta_s \geq \delta$ and that if the plan y, φ is followed then debt is never greater than $2d^*$. Just as in calculation 2) above, this implies that for every $t \leq T$:

$$d_{h_{t+1}} \leq \left[d_{h_t} + \frac{1}{\delta}(\bar{f}^i e_{h_t} - e^i_{h_t}) \right]^+ \left(\frac{e_{h_{t+1}}}{e_{h_t}} \right)^{1-g} \qquad (15)$$

Write

$$z_{h_t} = \frac{1}{\delta} \left[\bar{f}^i e_{h_t} - e^i_{h_t} \right] e^{g-1}_{h_t}$$

Multiply both sides of (15) by $e^{g-1}_{h_{t+1}}$, and keep in mind that debt is non-negative to obtain

$$0 \leq d_{h_{t+1}} e^{g-1}_{h_{t+1}} \leq [d_{h_t} {e_{h_t}}^{g-1} + z_{h_t}]^+$$

Set

$$\Psi = \{ t < T : d_{h_t} {e_{h_t}}^{g-1} + z_{h_t} < 0 \}$$

$$n^* = \begin{cases} \max_\Psi n & \text{if} \quad \Psi \neq \emptyset \\ -1 & \text{if} \quad \Psi = \emptyset \end{cases}$$

Note that $d_{h_{n^*+1}} = 0$ so, recalling the definitions of C and $\mathcal{H}(\delta)$ we find

$$\begin{aligned} d_{h_T} &\leq \sum_{t=n^*+1}^{T} z_t \\ &= \sum_{t=0}^{T} z_t - \sum_{t=0}^{n^*} z_t \\ &< \frac{\bar{\varepsilon}}{4(1-\delta)} m^{1-g} + \frac{\bar{\varepsilon}}{4(1-\delta)} m^{1-g} \\ &= \frac{\bar{\varepsilon}}{2(1-\delta)} m^{1-g} \end{aligned}$$

Hence $h \in \mathcal{H}_1$. Since h is arbitrary, it follows that $\mathcal{H}(\delta) \subset \mathcal{H}_1$, as asserted.

Combining the fact that $\mathcal{H}(\delta) \subset \mathcal{H}_1$ with the estimate (14), we obtain

$$\begin{aligned} U^i_\delta(y) \geq\ & \mathrm{Prob}[\mathcal{H}(\delta)] U^i_\delta(\overline{f^i} e - \bar{\varepsilon} e^{1-g}) \\ & - \delta^{T^*(\delta)} [u^i(M) - u^i(m - \bar{\varepsilon} M^{1-g})] \\ & - \{1 - \mathrm{Prob}[\mathcal{H}(\delta)]\} \, |u^i(m - \bar{\varepsilon} M^{1-g})| \end{aligned}$$

In view of the estimate (11), we obtain a lower bound on the utility of the consumption plan $\underline{f}^i e - \bar{\varepsilon} e^{1-g}$

$$U^i_\delta(\overline{f^i} e - \bar{\varepsilon} e) \geq U^i_\delta(\overline{f^i} e) - \frac{\eta}{4}$$

Hence Lemma 4 guarantees that, for δ sufficiently close to 1 we have

$$\begin{aligned} \mathrm{Prob}[\mathcal{H}(\delta)] U^i_\delta(\overline{f^i} e - \bar{\varepsilon} e^{1-g}) &\geq U^i_\delta(\overline{f^i} e) - \frac{\eta}{3} \\ \delta^{T^*(\delta)} [u^i(M) - u^i(m - \bar{\varepsilon} M^{1-g})] &\leq \frac{\eta}{3} \\ \{1 - \mathrm{Prob}[\mathcal{H}_0(T^*(\delta))]\} \, |u^i(m - \bar{\varepsilon} M^{1-g})| &\leq \frac{\eta}{3} \end{aligned}$$

whence

$$U^i_\delta(y) \geq U^i_\delta(\overline{f^i} e) - \eta$$

This is the desired lower bound on equilibrium utilities.

To establish an upper bound on equilibrium utilities, set

$$\alpha = \min_{i,s} Du^i(e_s) \ , \ \beta = \max_{i,s} Du^i(\overline{f^i} e_s)$$

Suppose there is a trader k for whom

$$U^k(x^k) > U^k(\overline{f}^k e) + N\left(\frac{\beta}{\alpha}\right)\eta$$

For each λ with $0 < \lambda < 1$ define $z^\lambda \in \ell^\infty(S)$ by

$$z_s^\lambda = \begin{cases} x_s^k & \text{if} \quad x^k \leq \underline{\overline{f}}^k e_s \\ \lambda x_s^k + (1-\lambda)\,\overline{f}^k e_s & \text{if} \quad x^k > \underline{f}^k e_s \end{cases}$$

There is a unique λ^* for which $U^k(z^{\lambda^*}) = U^k(\overline{f}^k e) + \eta$. Set $X^k = z^{\lambda^*}$, For each $j \neq k$, define X^j by:

$$X_s^j = \begin{cases} x_s^j & \text{if} \quad x^k \leq \underline{\overline{f}}^k e_s \\ \frac{1}{N}\left(x_s^k - X_s^k\right) + x_s^j & \text{if} \quad x^k > \underline{f}^k e_s \end{cases}$$

By definition, $U^k(X^k) = U^k(\overline{f}^k e) + \eta$. Our construction and the lower bound obtained previously guarantee that for $j \neq k$,

$$\begin{aligned} U^j(X^j) &\geq U^j(x^j) + \frac{1}{N}\left(\frac{\alpha}{\beta}\right)\left(U^k(x^k) - U^k(X^k)\right) \\ &> U^j(\overline{f}^j e) - \eta + \eta \\ &= U^j(\overline{f}^j e) \end{aligned}$$

Hence the allocation (X^i) Pareto dominates the allocation $(\overline{f}^i e)$. As we have noted previously, the allocation $(\overline{f}^i e)$ is Pareto optimal, so this is a contradiction. We conclude that

$$U^k(x^k) \leq U^k(\overline{f}^k e) + N\left(\frac{\beta}{\alpha}\right)\eta$$

for each trader k. This is the desired upper bound on utilities.

Combining our lower and upper bounds, we conclude that, if δ is sufficiently close to 1 then

$$U^i(\overline{f}^i e) - \eta < U^i(x^i) \leq U^i(\overline{f}^i e) + N\left(\frac{\beta}{\alpha}\right)\eta$$

for each i. Since $\eta > 0$ is arbitary, this yields (i).

To prove (ii), fix $\varepsilon > 0$. Pareto optimality of $(\overline{f}^i e)$ implies there are strictly positive welfare weights (λ^i), summing to 1, for which $(\overline{f}^i e)$ maximizes the weighted sum $\sum \lambda^i U^i(y^i)$ of utilities. Our assumptions guarantee that $(\overline{f^i}e_s)$ is Pareto optimal for each date-event s and hence maximizes the weighted sum $\sum \lambda^i u^i(y^i_s)$ with the same weights. Strict concavity of period utility functions implies that $(\overline{f^i}e_s)$ is the unique allocation maximizing this weighted sum. It follows from continuity of the weighted sum and finiteness of the range of the social endowment map that there is an $\bar{\varepsilon} > 0$ with the property that if (x^i_s) is any feasible allocation and $|\overline{f^i}e_s - x^i_s| > \varepsilon$ for some i, s then $\sum \lambda^i u^i(\overline{f^i}e_s) > \sum \lambda^i u^i(x^i_s) + \bar{\varepsilon}$. Hence if $\zeta = \left(q_\zeta, (x^i_\zeta), (\theta^i_\zeta)\right) \in EQ_\delta$ we obtain

$$\begin{aligned} \sum \lambda^i U^i(\overline{f^i}e) - \sum \lambda^i U^i(x^i_\zeta) &> (1-\delta) \sum_{s \in S^c_\varepsilon(\delta;\zeta)} \bar{\varepsilon}\delta^{t(s)}\pi_s \\ &= \bar{\varepsilon}(1-\delta) \sum_{s \in S^c_\varepsilon(\delta;\zeta)} \delta^{t(s)}\pi_s \qquad (16) \end{aligned}$$

In view of (i), the first expression of (16) tends to 0 as δ tends to 1; hence the last expression also tends to 0 as δ tends to 1. Because $\bar{\varepsilon}$ depends only on ε and is independent of δ. this implies (ii).

To prove (iii), fix $\varepsilon > 0$. Because the range of the endowment mapping is finite, we can choose $\varepsilon^* > 0$ sufficiently small that for every trader i and every date-event $s \in S$, if consumptions at s and at all date-events $\sigma \in s^+$ are within ε^* of perfect risk sharing, then marginal utilities are within ε of the marginal utilities at perfect risk sharing. Formally: if $s \in S$, if $|x_s - \overline{f^i}e_s| \le \varepsilon^*$ and if $|x_\sigma - \overline{f^i}e_\sigma| \le \varepsilon^*$ for every $\sigma \in s^+$ then

$$\left| \frac{Du^i(x_s)}{Du^i(x_\sigma)} - \frac{Du^i(\overline{f^i}e_s)}{Du^i(f^i e_\sigma)} \right| \le \varepsilon$$

Now fix an equilibrium $\zeta \in EQ_\delta$. The first order conditions for equilibrium imply that if $s \in S^p_\varepsilon(\delta;\zeta)$ then either (i) $s \in S^c_{\varepsilon^*}(\delta;\zeta)$, or (ii) $\sigma \in S^c_{\varepsilon^*}(\delta;\zeta)$ for some $\sigma \in s^+$. Hence

$$
\begin{aligned}
\sum_{s\in S^{p}_{\varepsilon}(\delta;\zeta)} \delta^{t(s)}\pi_s &\le \sum_{s\in S^{c}_{\varepsilon*}(\delta;\zeta)} \delta^{t(s)}\pi_s \\
&\quad + \sum_{\sigma\in S^{c}_{\varepsilon*}(\delta;\zeta)} \delta^{t(\sigma^-)}\pi_{\sigma^-} \\
&\le \sum_{s\in S^{c}_{\varepsilon*}(\delta;\zeta)} \delta^{t(s)}\pi_s \\
&\quad + \frac{1}{\delta}\sum_{\sigma\in S^{c}_{\varepsilon*}(\delta;\zeta)} \delta^{t(\sigma)}\frac{\pi_\sigma}{\pi(\sigma|s)} \\
&\le \sum_{s\in S^{c}_{\varepsilon*}(\delta;\zeta)} \delta^{t(s)}\pi_s \\
&\quad + \frac{1}{\delta\min_{s,\sigma}\pi(\sigma|s)}\sum_{\sigma\in S^{c}_{\varepsilon*}(\delta;\zeta)} \delta^{t(\sigma)}\pi_\sigma
\end{aligned}
$$

Multiplying by $1-\delta$ and applying (ii) yields (iii). ■

As we have commented earlier, our result is driven by the ability of individuals to borrow and not by their ability to save. The distinction is important because there is an asymmetry in our ability to estimate the prices of riskless or risky assets. In order to borrow, individuals must sell bonds, so individuals who smooth consumption by borrowing desire high asset prices (low interest rates); Lemmas 1 and 2 provide the bounds we need. In order to save, however, individuals must *buy* bonds, so individuals who smooth consumption by saving desire low asset prices (high interest rates). Unfortunately, we do not know how to obtain *a priori* upper bounds on asset prices (lower bounds on interest rates). Indeed, even when there is no aggregate risk, it seems possible that equilibrium interest rates could be arbitrarily negative at some date-events. (Our assumptions on preferences and endowments guarantee that equilibrium interest rates must equal subjective interest rates when markets are complete, but do not rule out negative interest rates when — as here — markets are incomplete.)

We have assumed here that traders are equally patient (that is, share a common discount factor); it is not entirely clear what conclusions would obtain if traders are unequally patient. The problem is that the lower bound for the equilibrium utility of the *most* patient trader depends on the estimated stopping time of the alternative plan, this estimated stopping time depends on the lower bound for equilibrium prices, and this lower bound depends in turn on the discount factor of the *least* patient trader. If the most patient trader is much more patient than the least patient trader, utility accumulated before this stopping time might represent an insignificant portion of the lifetime utility of the most patient trader. Thus the argument given does not generalize unless discount factors are sufficiently similar. On the other hand, it seems natural to suppose that, at equilibrium, the least patient traders consume more in early date-events and less (or nothing) in later date-events, which suggests that sharper price estimates might be available.

Appendix 2

Choose $\varepsilon > 0$ sufficiently small that

$$v = \frac{1}{2}\left[H^{1-\gamma} - \frac{L^{1-\gamma}}{(1-k\varepsilon)^{\gamma}}\right] > 0 \text{ and } (1-2k\varepsilon)\frac{H}{L} > 1$$

We show that for ε at least this small, and δ sufficiently close to 1, *no* equilibrium is close to perfect risk sharing. Indeed, we provide an explicit bound on the utility loss compared to perfect risk sharing.

The argument depends on two bounds: an absolute upper bound (independent of the discount factor) on the equilibrium debt of the small trader at any date-event, and an absolute upper bound (independent of the discount factor) on the equilibrium savings of the small trader on a set of date-events of positive time-discounted probability. Parts I and II below derive these bounds, and Part III uses these bounds to provide the desired estimate of the utility loss compared to perfect risk sharing.

Part I: Debt

Fix a discount factor δ and an equilibrium of $\mathcal{E}_{\delta}$. For each date-event s, write $1 - f_s, f_s$ for the endowment shares of the large and small traders (respectively), so that $(1 - f_s)e_s, f_s e_s$ are the endowments of the large and small traders. Write x_s, y_s for the equilibrium consumptions of the large and small traders. Write d_s for the debt of the small trader, so that $-d_s$ is the savings of the small trader.

Choose α so that

$$1 < \alpha < \frac{(1-2k\varepsilon)H}{L}$$

and choose V sufficiently large that

$$\left[\frac{\alpha^{\gamma}-1}{\alpha^{\gamma}+}\right]V - \frac{Lk\varepsilon}{[L(1-k\varepsilon)]^{\gamma}} > v$$

We assert that, independently of the discount factor δ, the debt of the small trader never exceeds $VH^{\gamma} + k\varepsilon$. The argument is in several steps.

Step 1 Although we wish to bound d_s, which is debt at the beginning of date-event s, it is convenient to work with $D_s = d_s + y_s - f_s e_s$, which is debt at the end of date-event s and $W_s = D_s x_s^{-\gamma}$, which might be thought of as potential debt. We show that potential debt is bounded (independently of the discount factor δ); this will yield a bound on debt.

Step 2 Note that $x_s > 0$ at every date-event s. For, if not, the consumption plan x' defined by $x'_s = t x_s + (1-t)(1-f_s)e_s$ would be feasible for every t with $0 < t < 1$ and would give greater utility than the equilibrium consumption plan x for t sufficiently close to 1. Hence the equilbrium bond price is determined by the marginal utilities of the large trader. Writing $s^+ = \{A,B,C,D\}$, this means that

$$q_s = \delta \frac{x_A^{-\gamma} + x_B^{-\gamma} + x_C^{-\gamma} + x_D^{-\gamma}}{4 x_s^{-\gamma}}$$

Step 3 We now describe the evolution of debt. Fix $s \in S$. The small trader enters s with debt d_s, consumes y_s, and finances his plan by selling endowment and buying or selling the riskless security. His debt entering any succeeding date-event $\sigma \in s^+$ is therefore

$$d_\sigma = \frac{1}{q_s}[d_s + y_s - f_s e_s]$$

Keeping in mind that $x_s + y_s = e_s$ and the definition of D_σ, D_s yields

$$\begin{aligned} D_\sigma &= d_\sigma + y_\sigma - f_\sigma e_\sigma \\ &= \frac{1}{q_s}[d_s + y_s - f_s e_s] + y_\sigma - f_\sigma e_\sigma \\ &= \frac{1}{q_s} D_s + (e_\sigma - x_\sigma) - f_\sigma e_\sigma \\ &= \frac{1}{q_s} D_s + [(1-f_\sigma)e_\sigma - x_\sigma] \end{aligned}$$

Since $D_s = x_s^\gamma W_s$ and $D_\sigma = x_\sigma^\gamma W_\sigma$, we obtain

$$x_\sigma^\gamma W_\sigma = \frac{1}{\delta} x_s^\gamma W_s \frac{4 x_s^{-\gamma}}{x_A^{-\gamma} + x_B^{-\gamma} + x_C^{-\gamma} + x_D^{-\gamma}} + [(1-f_\sigma)e_\sigma - x_\sigma]$$

so

$$W_\sigma = \frac{1}{\delta} W_s \frac{4}{(\frac{x_\sigma}{x_A})^\gamma + (\frac{x_\sigma}{x_B})^\gamma + (\frac{x_\sigma}{x_C})^\gamma + (\frac{x_\sigma}{x_D})^\gamma} \; + \; \frac{1}{x_\sigma^\gamma}[(1-f_\sigma)e_\sigma - x_\sigma]$$

Step 4 We establish the following claim:

Claim If $W_s > V$ then $\exists \sigma \in s^+$ such that $W_\sigma - W_s > v$.

To see this, recall first that, by construction, the social endowment is small in two date-events in s^+ and large in two; say $e_A = e_C = L, e_B = e_D = H$. Without loss, assume $x_A \leq x_C$ and $x_B \leq x_D$. Then

$$\begin{aligned} W_A &\geq W_s \frac{2}{(\frac{x_A}{x_A})^\gamma + (\frac{x_A}{x_B})^\gamma} + \frac{1}{x_A^\gamma}[(1-f_A)e_A - x_A] \\ W_B &\geq W_s \frac{2}{(\frac{x_B}{x_A})^\gamma + (\frac{x_B}{x_B})^\gamma} + \frac{1}{x_B^\gamma}[(1-f_B)e_B - x_B] \end{aligned}$$

and hence

$$W_A - W_s \geq W_s \frac{1-(\frac{x_A}{x_B})^\gamma}{1+(\frac{x_A}{x_B})^\gamma} + \frac{1}{x_A^\gamma}[(1-f_A)e_A - x_A]$$

$$W_B - W_s \geq W_s \frac{1-(\frac{x_B}{x_A})^\gamma}{1+(\frac{x_B}{x_A})^\gamma} + \frac{1}{x_B^\gamma}[(1-f_B)e_B - x_B]$$

If $x_A/x_B \geq \alpha$, then $W_B - W_s \geq v$. Assume therefore that $x_A/x_B < \alpha$. Notice that

$$\frac{1-(\frac{x_B}{x_A})^\gamma}{1+(\frac{x_B}{x_A})^\gamma} = -\frac{1-(\frac{x_A}{x_B})^\gamma}{1+(\frac{x_A}{x_B})^\gamma}$$

Keeping this in mind, and making use of our choice of α and the assumption that $x_A/x_B < \alpha$, a little algebra leads to the following inequalities:

$$W_A - W_s \geq +W_s \frac{1-(\frac{x_A}{x_B})^\gamma}{1+(\frac{x_A}{x_B})^\gamma} - \frac{Lk\varepsilon}{[L(1-k\varepsilon)]^\gamma} \tag{17}$$

$$W_B - W_s \geq -W_s \frac{1-(\frac{x_A}{x_B})^\gamma}{1+(\frac{x_A}{x_B})^\gamma} + \frac{(1-k\varepsilon)H - \alpha L}{H^\gamma} \tag{18}$$

Adding (17) and (18) yields

$$\begin{aligned} (W_B - W_s) + (W_A - W_s) &\geq \frac{(1-k\varepsilon)H - \alpha L}{H^\gamma} - \frac{Lk\varepsilon}{[L(1-k\varepsilon)]^\gamma} \\ &> k\varepsilon \left[H^{1-\gamma} - \frac{L^{1-\gamma}}{(1-k\varepsilon)^\gamma} \right] \\ &> 2v \end{aligned}$$

In particular, either $W_A - W_s > v$ or $W_B - W_s > v$, as desired.

Step 5 We assert that potential debt is bounded by V. To see this, suppose not. In view of Step 4, there is a sequence of successive date-events along which potential debt tends to infinity. Hence either there is a sequence of successive date-events along which consumption of the large trader tends to 0 and debt of the small trader is non-negative, or there is a sequence of (not necessarily successive) date-events along which end of period debt of the small trader tends to infinity. The first alternative is untenable: If consumption of the large trader is small at date-event s then marginal utility for consumption is large. Since the debt of the small trader is positive, the savings of the large trader must also be positive. Hence, for sufficiently small $\lambda > 0$, the large trader would find it feasible and preferable to alter his consumption plan at s and all succeeding date events to $(\lambda x_s + (1-\lambda)(1-f_s)e_s)$; this would contradict optimality. On the other hand, the second alternative is also untenable: since endowments are bounded, if end of period debt tends to infinity then beginning of period debt must also tend to infinity. But Levine and Zame (1994) show that, at any equilibrium, debt is bounded.[21] We conclude that potential debt is bounded by V.

Step 6 It follows immediately from the definition of potential debt that debt of the small trader is bounded by $VH^{\gamma} + k\varepsilon$. This completes Part I.

Part II: Savings

We choose $\nu < 1$ and show that savings of the small trader $-d_s$ are bounded by $k\varepsilon H/(1-\nu)$ at a set of date-events of time-discounted probability bounded away from 0 (independently of δ). Again, the argument is in several steps.

Step 1 Write

$$\begin{aligned} q_L &= \delta\frac{H^{-\gamma}+L^{-\gamma}}{2L^{-\gamma}} \\ q_H &= \delta\frac{H^{-\gamma}+L^{-\gamma}}{2H^{-\gamma}} \end{aligned}$$

[21]In general, the bound on debt established by Levine and Zame (1994) will depend on the discount factor δ. The present argument is subtle because we require a bound on debt that is independent of the discount factor.

Arguing exactly as in the proof of Lemma 4, we see that the price of the riskless bond satisfies:

$$q_s \geq \begin{cases} q_L & \text{if} \quad e_s = L \\ q_H & \text{if} \quad e_s = H \end{cases} \tag{19}$$

Note that $q_L q_H > 1$ if δ is sufficiently close to 1. We may therefore choose and fix real numbers $\chi, \nu < 1$ such that

$$q_L^k q_H^\ell > \frac{1}{\nu^{k+\ell}} \quad \text{if} \quad \frac{k}{\ell} > \chi$$

for all δ sufficiently close to 1.

Step 2 As noted earlier, the evolution equation for the debt of the small trader is

$$d_\sigma = \frac{1}{q_s}[d_s - f_s e_s + y_s]$$

for any $\sigma \in s^+$. Because the initial debt and saving are 0, we can bound savings at later date-events: for every date T and every date-event s with $t(s) = T + 1$ we have

$$-d_s \leq \sum_{t=0}^{T} \left(f_{s_t} e_{s_t} \prod_{t'=t}^{T} \frac{1}{q_{s_{t'}}} \right)$$

In view of the price estimate (19),

$$\prod_{t'=t}^{T} \frac{1}{q_{s_{t'}}} \leq \frac{1}{q_H^{k(t')} q_L^{\ell(t')}}$$

where $k(t)$ is the number of times in the specified range that $e_s = H$ and $\ell(t) = T - t$ is the number of times that $e_s = L$. Thus, if $k(t)/\ell(t) > \chi$ then

$$\prod_{t'=t}^{T} \frac{1}{q_{s_{t'}}} \leq \nu^{T-t}$$

Hence if $k(t)/\ell(t) > \chi$ for every $t \leq T$ then

$$-d_s \leq \sum_{t=0}^{T} \left(k\varepsilon H \nu^{T-t} \right) \leq k\varepsilon H \frac{1}{1-\nu}$$

Step 3 For each T, consider the set $G(T)$ of date-events s with $t(s) = T+1$ for which $k(t)/\ell(t) > \chi$ for every $t \leq T$, and write G for the set of date-events s for which $s \in G(t(s)+1)$. Because the underlying Markov process is i.i.d. with transition probabilities equal to 1/4, the usual coin-tossing inequalities guarantee that we can choose a real number $\beta > 0$ independent of T such that $\text{Prob}(G(T)) > \beta$. It follows that the time-discounted probability of G also exceeds β, independently of the discount factor δ.

Part III: Utility

Let K be an integer sufficiently large that

$$K\varepsilon - \frac{k\varepsilon H}{1-\nu} > VH^{\gamma} + k\varepsilon$$

Write P for the set of date-events s for which bond prices in the date-events $s, s^{+}, s^{+2}, \ldots, s^{+K}$ are below 1 whenever the social endowment is low. Arguing exactly as in the proof of Theorem B, we see that if δ is sufficiently close to 1, then the time-discounted probability of P exceeds $1 - \frac{1}{2}\beta$. Because the time-discounted probability of G exceeds β, it follows that the time-discounted probability of $P \cap G$ exceeds $\frac{1}{2}\beta$.

For $s \in P \cap G$, set $s_1 = s$; let $s_2 \in s_1^{+}$ be the unique date-event in which social endowment and small trader endowment are both small, let $s_3 \in s_2^{+}$ be the unique date-event in which social endowment and small trader endowment are both small, and so forth. The sequence of date-events $s_1, s_2, \ldots s_{K+1}$ has these properties:

(i) $-d_{s_1} \leq \frac{k\varepsilon H}{1-\nu}$

(ii) at each s_k the underlying Markov process is in state 1 (low social endowment, small share for small trader)

(iii) at each s_k bond prices are bounded by 1: $q_{s_k} \leq 1$

Suppose that $y_{s_k} - \varepsilon L \geq \varepsilon$ for each $k \leq K+1$. It follows from (ii) that the small trader dissaves by at least the fixed amount ε at each date-event s_k. In view of (iii), debt must grow by at least the fixed amount ε at each date-event s_k. Hence, making use of (i) yields

$$d_{s_{K+1}} \geq K\varepsilon + d_{s_1} \geq K\varepsilon - \frac{k\varepsilon H}{1-\nu} > VH^{\gamma} + k\varepsilon$$

However, we have shown that the debt of the small trader never exceeds $VH^{\gamma}+k\varepsilon$, so this is a contradiction.

We conclude that for each $s \in P \cap G$ there is a date event $\sigma \geq s$ such that $t(\sigma) - t(s) \leq K$ and $y_{\sigma} - \varepsilon L < \varepsilon$. Perfect risk sharing requires the small trader to consume at least $\frac{k+1}{2}\varepsilon L > 2\varepsilon L$ in every date-event, so this represents an (unweighted) utility loss (compared to perfect risk sharing) of at least $[\varepsilon L][Du^2(2\varepsilon L)]$. Because $P \cap G$ is a set of date-events of time-discounted probability at least $\beta/2$, taking into account the possibility of counting some date-events more than once, it follows that the total utility loss compared to perfect risk sharing is at least $[\frac{1}{2K}\beta\delta^K][\varepsilon L][Du^2(2\varepsilon L)]$.

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