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Intertemporal Separability in Overlapping-Generations Models*

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In a pure exchange overlapping-generations model with many goods, but a single consumer with preferences separable between two periods of life, there are (generically) finitely many equilibria in which money has no value. If money has value, then (generically) there is at most one dimension of indeterminacy. This property does not generalize to a model with many consumers and general preferences. It is shown why a separable representative consumer implies such strong conclusions. It is also shown that the absence of income effects leads to similar results. *Journal of Economic Literature* Classification Numbers: 021, 023, 111. © 1984 Academic Press, Inc.

1. INTRODUCTION

Gale [4] has shown that in a pure exchange overlapping-generations economy with two-period lived consumers and a single consumption good there are generically a finite number of perfect foresight equilibria along which money has no value. If money has value, then there is (generically) at most one dimension of indeterminacy, which can be indexed by the price of money relative to that of the consumption good. We refer to this as the *one-good result*. In recent years a number of extensions of the one-good result have been put forth. Balasko and Shell [1] have argued that the result holds if there are many goods in each of the two periods, but a single representative consumer with Cobb–Douglas preferences in each generation. Brown

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and Geanakopolos [2], using non-standard analysis, and Geanakopolos and Polemarchakis [5] have demonstrated that a single consumer with preferences separable between the two periods of his life leads to the same result, at least in the case where money is useless.

The extent to which these results further generalize has been explored by Kehoe and Levine [8], who report a robust example with a single consumption good and a representative individual with additively separable (CES) preferences who lives for three periods yet exhibits indeterminacy when money has no value, and more than one dimension of indeterminacy when it does. We are led in this paper to ask what special features of separable preferences in the two-period case lead to the one-good result.

Kehoe and Levine [7] have outlined a general approach for analyzing the determinacy of equilibria locally near a steady state of a generic economy. Unfortunately, economies with a single separable consumer are degenerate because the matrix of derivatives of young people's excess demand with respect to prices when they are old is singular. In this paper we extend the previous analysis to cover this case and show that the one-good result holds locally near the steady state of a stationary economy in which two-period lived consumers are "almost" identical and have "almost" separable preferences. Thus, although the Kehoe and Levine [7] paper has shown that there are open sets of economies with any degree of indeterminacy no greater than number of goods, this paper shows why the work of Balasko, Shell, Brown, Geanakopolos, and Polemarchakis has identified an open set of economies in which the one-good result holds.

Unfortunately, separability combined with a representative consumer is not plausible when combined with only two periods of life. Typically we would expect at any moment of time many quite dissimilar individuals due simply to differences in age. A possibly more palatable assumption that leads to results similar to the one-good result is that there are no large income effects. We discuss this in the concluding section.

2. The Overlapping-Generations Model

Each generation $t \geqslant 1$ is identical and lives in periods t and $t+1$. There are n goods in each period. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ when young and $z(p_t, p_{t+1})$ when old. The vector $p_t = (p_t^1, ..., p_t^n)$ denotes prices in period t. Intertemporal trade is possible, so the aggregate budget constraint (Walras's law) has the form $p_t' y(p_t, p_{t+1}) + p_{t+1}' z(p_t, p_{t+1}) = 0$. In addition, excess demand is homogeneous of degree zero in prices (p_t, p_{t+1}). We also assume excess demand is continuously differentiable, which, as Debreu [3] and Mas-Colell

[10] have shown, entails little loss of generality. An equilibrium price path for this economy is one in which excess demand vanishes in each period: $z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0$ for $t > 1$ and $z_0(p_1) + y(p_1, p_2) = 0$ for $t = 1$, where z_0 is the excess demand of old consumers in period one. There are two types of equilibria in the economy. To distinguish between them we set $\mu = p_1' z_0(p_1)$. Iterated application of the equilibrium condition and Walras's law shows that $-p_t' y(p_t, p_{t+1}) = p_{t+1}' z(p_t, p_{t+1}) = \mu$ at all times. Thus μ is the fixed nominal savings of young people in each period or, equivalently, the fixed stock of fiat money (at least if $\mu \geqslant 0$). If $\mu = 0$, we call this a *real* path; if $\mu \neq 0$, we call it a *nominal* path. A *steady state* of the economy is a relative price vector p and inflation factor γ such that $p_t = \gamma^t p$ is an equilibrium of the economy. The steady state rate of interest is $1/\gamma - 1$.

In the remainder of this section we summarize some general results from Kehoe and Levine [7]. If a steady state is nominal, then Walras's law and the equilibrium condition imply that $\gamma = 1$. Conversely, if a steady state is real, then generically $\gamma \neq 1$. We focus on the behavior of paths near a steady state. In particular, we ask how many equilibrium paths converge to the steady state. The stable manifold theorem from the theory of dynamical systems described in Irwin [6] implies that generically this question can be answered by linearizing the equilibrium conditions. Making use of the fact that derivatives of excess demand are homogeneous of degree minus one, we can write the linearized system near a steady state (p, γ) as

$$D_2 y\, p_{t+1} + (D_1 y + \gamma D_2 z) p_t + \gamma D_1 z\, p_{t-1} = 0, \qquad t \geqslant 1 \tag{1}$$

$$D_2 y\, p_2 + (D_1 y + D z_0) p_1 = D z_0 p - z_0 - y \tag{2}$$

where $D_1 y$ is, for example, the matrix of partial derivatives of y with respect to its first vector of arguments and where all functions and their derivatives are evaluated at $(p, \gamma p)$.

In the generic case $D_2 y$ is non-singular, and (1) can be solved to yield a second order difference equation. Define the characteristic matrix $R(\phi)$ by the rule

$$R(\phi) = D_2 y\, \phi^2 + (D_1 y + \gamma D_2 z)\phi + \gamma D_1 z. \tag{3}$$

The characteristic values of the system are the roots of the equation $\det R(\phi_i) = 0$, and, if the vectors f_i satisfy $R(\phi_i) f_i = 0$ and $f_i \neq 0$, the characteristic vectors of the system are $(f_i, \phi_i f_i)$. Homogeneity implies that $\phi = \gamma$ is a root of R with $R(\gamma) p = 0$ where p is the steady state vector of relative prices. Generically, this is the only root on the circle of radius γ in the complex plane.

We consider a nominal steady state with $\mu \neq 0$ and $\gamma = 1$ first. In equilibrium $p_1' y(p_1, p_2) = -\mu$, which can be linearized at the steady state as

$$p' D_2 y p_2 + (y' + p' D_1 y) p_1 = -\mu. \tag{4}$$

This defines a $2n - 1$ dimensional subspace of the initial conditions (p_1, p_2), which is invariant since μ is constant on paths. It contains all $2n - 1$ characteristic vectors of the system except the vector (p, p) that corresponds to the root $\gamma = 1$. Let n^s be the number of roots inside the unit circle. The corresponding n^s dimensional subspace spanned by characteristic vectors is the space of initial conditions (p_1, p_2) that yield paths that converge to the steady state. Condition (2) defines an n dimensional subspace of the $2n - 1$ dimensional space of vectors that satisfy (4). The intersection of the two spaces generically has dimension $n^s + n - (2n - 1) = n^s - n + 1$. Thus, generically, there is an $n^s - n + 1$ dimensional set of equilibria. If $n^s < n - 1$, the set is empty; if $n^s = n - 1$, the equilibrium is unique. Notice, however, that in the nonlinear system this implies only that there is a unique equilibrium in a sufficiently small neighborhood of the steady state—no implication of global uniqueness follows.

At real steady states the price level is indeterminate, so we work with prices in a $2n - 1$ dimensional space of normalized prices, throwing out the characteristic value γ associated with characteristic vector $(p, \gamma p)$. It can be shown that the condition for stability in this lower dimensional system is that characteristic values be less than γ is modulus. Since we consider only initial conditions with $\mu = 0$, condition (4) further reduces the dimension of the system to $2n - 2$. It can be shown using Walras's law that the eigenvalue thrown out in this reduction is equal to 1 and that this root governs the behavior of paths with nominal initial conditions near a real steady state: If $\gamma > 1$, then asymptotically money does not matter; if $\gamma < 1$, then initial conditions with valued money cannot yield equilibrium price paths that approach the steady state. Let the number of remaining eigenvalues that lie inside the circle of radius γ be $\bar{n}^s$. The initial condition (2) defines an $n - 1$ dimensional space and thus the dimension of equilibria that converge to the real steady state is $\bar{n}^s + (n - 1) - (2n - 2) = \bar{n}^s - n + 1$.

3. Implications of a Single Separable Consumer

Kehoe and Levine [9] prove that there are robust examples of economies with any value of $0 \leqslant n^s \leqslant 2n - 1$ and $0 \leqslant \bar{n}^s \leqslant 2n - 2$. A case of particular interest is when the characteristic values *split*, that is, $n - 1$ lie inside, and $n - 1$ outside, the circle of radius γ. In this case $\bar{n}^s = n - 1$, which implies that real paths are locally unique, and $n - 1 \leqslant n^s \leqslant n$, which implies that nominal paths are either locally unique or have a single dimension of indeterminacy. Thus the one-good result holds locally near steady states if and only if the system splits. In the extreme case where $n-1$ values lie outside the circle of radius γ and $n-1$ values are exactly equal to zero, we say that the system *splits exactly*. In this case $n-1$ prices are thrown out as unstable and $n-1$ jump directly to their steady state value. The dynamical system really only involves two prices: one price for future consumption and one for current consumption. Thus, a system that splits exactly exhibits the same dynamic behavior as a model with one good in each period. In the real case we get to throw out one of these prices as corresponding to paths with valued fiat money, and we see that the system jumps directly to the steady state. In the nominal case there is one eigenvalue that is not determined. If it lies inside the unit circle, there is one dimension of indeterminacy, but it is always possible to jump right to the steady state. If it lies outside the unit circle, there is a uniquely determined path going to, but not generally equal to, the steady state.

The results of Balasko, Shell, Brown, Geanakopolos and Polemarchakis in our local context should imply that if there is a single separable consumer in each generation the system splits. Suppose the representative consumer has utility $u(y, z)$ for net trades. The consumer maximizes $u(y, z)$ subject to the budget constraint $p_t' y + p_{t+1}' z = 0$. Assuming that the utility function has all the necessary properties, we can characterize the solution to this problem by the usual first order conditions:

$$\begin{aligned} D_1 u - \lambda p_t' &= 0 \\ D_2 u - \lambda p_{t+1}' &= 0 \\ p_t' y + p_{t+1}' z &= 0 \end{aligned} \tag{5}$$

for some $\lambda > 0$. Using the implicit function theorem, we can compute the partial derivatives of y and z by differentiating (5). For example, after some tedious algebra we find that

$$D_2 y = -D_{11}^2 u^{-1} \left(D_{12}^2 u A \left(\lambda I - \frac{1}{b} cd' \right) + \frac{1}{b} p_t d' \right) \tag{6}$$

where

$$\begin{aligned} A &= (D_{22}^2 u - D_{21}^2 u \, D_{11}^2 u^{-1} \, D_{12}^2 u)^{-1} \\ b &= [p_t' \, p_{t+1}'] \begin{bmatrix} D_{11}^2 u & D_{12}^2 u \\ D_{12}^2 u & D_{22}^2 u \end{bmatrix}^{-1} \begin{bmatrix} p_t \\ p_{t+1} \end{bmatrix} \\ c &= p_{t+1} - D_{21}^2 u \, p_t \\ d &= \lambda A c + z(p_t, p_{t+1}). \end{aligned} \tag{7}$$

With one consumer with separable preferences, utility depends only on an index of consumption each period, so $u(y, z) = v(h(y), g(z))$ and thus

$$D_{12}^2 u = D_{12}^2 v \, Dh \, Dg'. \tag{8}$$

Furthermore, at the optimum, $Dh = \lambda p_t$ and $Dg' = \lambda p'_{t+1}$. Therefore

$$D_{12}^2 u = \lambda^2 D_{12}^2 v \, p_t p'_{t+1}. \tag{9}$$

Using (6), we see that

$$D_2 y = -D_{11} u^{-1} p_t \left(\lambda^2 D_{12}^2 v \, p'_{t+1} A \left(\lambda I - \frac{1}{b} cd' \right) + \frac{1}{b} d' \right) \tag{10}$$

has rank one, and similarly for $D_1 z$. We summarize our arguments with the following theorem.

THEOREM 1. *Intertemporally separable utility implies that both $D_2 y$ and $D_1 z$ have at most rank one.*

Now we examine the dynamics when $D_2 y$ and $D_1 z$ have rank one. To do so we use three regularity assumptions, each assumed to hold at all steady states. Using the topology and methods of Kehoe and Levine [9], we can show that these assumptions are generic in the space of economies having one consumer with separable preferences. Here we merely indicate why they are not particularly restrictive.

(R.1) $D_1 y + \gamma D_2 z$ and $D_2 z$ are non-singular.

In fact $D_1 y$ and $D_2 z$ are each the sum of a negative definite substitution matrix and a rank one income effects matrix. Since neither y nor z is generally homogeneous in a subset of prices, neither $D_1 y$, nor $D_2 z$, nor their sum is generally singular.

(R.2) $D_2 y (D_1 y + \gamma D_2 z)^{-1} D_2 y \neq 0.$

Recall that $D_2 y$ has proportional rows and proportional columns. Consequently (R.2) says that a column of $D_2 y$ under the linear mapping $(D_1 y + \gamma D_2 z)^{-1}$ should not be orthogonal to a row of $D_2 y$. There is no economic reason why it should be.

(R.3) There exists ϕ_2, not equal to either 0 or γ, with $\det R(\phi_2) = 0$.

At a real steady state generically $\gamma \neq 1$ and $R(1) = 0$, so $\phi_2 = 1$ will do. At a nominal steady state we are asserting the existence of one free eigenvalue. When $n = 1$, for example, we require that the two roots of R not both be 1.

The key technical result characterizes $R(\phi)$ under (R.1)–(R.3).

THEOREM 2. *Under* (R.1)–(R.3) *if $D_2 y$ and $D_1 z$ have rank one then*

(a) det $R(\phi)$ *is a polynomial of degree $n + 1$ with $n - 1$ roots equal to zero and the remaining roots $\phi_1 = \gamma$ and ϕ_2 given in* (R.3). *There are $n + 1$ vectors f_i such that $R(\phi_i) f_i = 0$ (where $\phi_i = 0$ for $i > 2$) and $(f_i, \phi_i f_i)$ are linearly independent.*

(b) *The linearized equilibrium condition* (1) *can be solved for p_{t+1} if and only if $(p_{t-1}, p_t) \in \langle (f_i, \phi_i f_i) \rangle$ (where $\langle\ \rangle$ means the space spanned by).*

(c) *If $(p_{t-1}, p_t) \in \langle (f_i, \phi_i f_i) \rangle$ then there is a unique value of p^{t+1} that solves* (1) *and satisfies $(p_t, p_{t+1}) \in \langle (f_i, \phi_i f_i) \rangle$.*

Proof. Observe first that if (ϕ_i, f_i) satisfies $R(\phi_i) f_i = 0$ and if $(p_{t-1}, p_t) = \sum_i \alpha_i (f_i, \phi_i f_i)$ then $(p_t, p_{t+1}) = \sum_i \alpha_i (\phi_i f_i, \phi_i^2 f_i)$ solves (1). Thus, if Q_L is the linear space of (p_{t-1}, p_t) that satisfies (1), then $\langle (f_i, \phi_i f_i) \rangle \subset Q_L$. Since (1) can be rewritten as

$$p_t = (D_1 y + \gamma D_2 z)^{-1} (D_2 y\, p_{t+1} + D_1 z\, p_{t-1}) \tag{11}$$

and since $D_2 y \langle p_{t+1} \rangle$ is one dimensional, it follows that $\dim(Q_L) = n + 1$.

To establish (b) we need only establish the existence of $n + 1$ linearly independent solutions $(f_i, \phi_i f_i)$. Obviously $\phi_1 = \gamma$ and $f_1 = p$; ϕ_2 is given by (R.3) and f_2 is any non-zero vector in the null space of $R(\phi_2)$. Finally, $R(0) = D_1 z$ has rank one, so there are $n - 1$ independent vectors $f_3, \ldots, f_{n+1}$ in its null space. This gives us $n + 1$ vectors. If they are dependent, then there are non-zero weights $\alpha_1, \ldots, \alpha_{n+1}$ such that

$$\begin{gathered} \sum_{i=1}^{n+1} \alpha_i f_i = 0 \\ \alpha_1 \phi_1 f_1 + \alpha_2 \phi_2 f_2 = 0. \end{gathered} \tag{12}$$

Since $\phi_2 \neq 0$, the second equation implies $\alpha_2 f_2 = -\alpha_1 (\phi_1 / \phi_2) f_1$ and, consequently, the first equation becomes

$$\alpha_1 f_1 (1 - \phi_1 / \phi_2) + \sum_{i=3}^{n+1} \alpha_i f_i = 0. \tag{13}$$

Since $\phi_1 \neq \phi_2$ by (R.3), this means f_1 is in the null space of $D_1 z$; or, since $f_1 = p$, that $D_1 z\, p = 0$. By homogeneity $D_1 z\, p + \gamma D_2 z\, p = 0$, so $D_2 z\, p = 0$, which contradicts $D_2 z$ being non-singular by (R.1). This establishes (b).

Next we establish (c). Let $(p_{t-1}, p_t) \in Q_L$. Part (b) of Theorem 2 implies that $(p_{t-1}, p_t) = \sum_i \alpha_i (f_i, \phi_i f_i)$ for unique weights α_i. Since $R(\phi_i) f_i = 0$, $p_{t+1} = \sum_i \alpha_i \phi_i^2 f_i$ solves (1) and, since $(p_t, p_{t+1}) = \sum_i \alpha_i \phi_i (f_i, \phi_i f_i)$ $(p_t, p_{t+1}) \in Q_L$. Therefore, a solution in Q_L exists. Since $D_2 y$ has rank one, any other $\tilde{p}_{t+1}$ that solves (1) must have the form $\tilde{p}_{t+1} = p_{t+1} + \eta$ where $D_2 y \eta = 0$. If $(p_t, p_{t+1} + \eta) \in Q_L$ then $(0, \eta) \in Q_L$ since $(p_t, p_{t+1}) \in Q_L$ already. Condition (1) implies that there is a vector x such that $D_2 y x = (D_1 y + \gamma D_2 z) \eta$. Consequently, (R.1) and $D_2 y \eta = 0$ imply that

$$D_2 y (D_1 y + \gamma D_2 z)^{-1} D_2 y x = 0 \tag{14}$$

and, if $\eta \neq 0$, $D_2 y x \neq 0$. Since $D_2 y$ has rank one this implies $D_2 y (D_1 y + \gamma D_2 z)^{-1} D_2 y = 0$, which contradicts (R.2). Therefore $\eta = 0$, and p_{t+1} is the unique solution in Q_L. Notice that if we use $(f_i, \phi_i f_i)$ as a basis for Q_L then $q_{t+1} = \mathrm{diag}(\phi_i)\, q_t$ so the linearized system satisfies the one-good result.

Finally, we prove (a). Clearly 0 is a root of $\det R(\phi)$; we first show that its algebraic multiplicity is at least $n-1$. To do so let f_{n+2} be any vector not in the null space of $D_1 z$. Define the matrix A to have columns $f_3, \ldots, f_{n+2}$; that is, the null vectors of $D_1 z$ plus one other independent vector. Since A has independent columns it is non-singular and $\det R(\phi) = 0$ if and only if $\det R(\phi) A = 0$. Writing this out, we see that

$$\det R(\phi) A = \det(D_2 y A \phi^2 + (D_1 y + \gamma D_2 z) A \phi + C) \tag{15}$$

where the first $n-1$ columns of C are zero and the last is $D_1 z f_{n+2}$. But then each of the first $n-1$ columns of $R(\phi) A$ contains a factor of ϕ implying $\det R(\phi) A = \phi^{n-1} \det \tilde{C}$ where $\tilde{C}$ has the first $n-1$ columns of $D_2 y A \phi + (D_1 y + \gamma D_2 z) A$ and the last column of $R(\phi) A$. So $\det \tilde{C}$ is a polynomial in (positive powers of) ϕ and the algebraic multiplicity of the zero root is at least $n-1$.

On the one hand, since $\det R(\phi)$ also has two distinct non-zero roots ϕ_1 and ϕ_2, it has degree at least $n+1$. On the other hand, we consider the backward characteristic matrix $B(\beta) \equiv \beta^2 R(\beta^{-1})$, which corresponds to running the system backward in time. Since $D_2 y$ has rank one, this implies $\det B(\beta)$ has at least $n-1$ zero roots. Factoring the forward and backward polynomials we see that each zero root of the backward polynomial reduces the degree of the forward polynomial by one and, therefore, that $\det R(\phi)$ has at most degree $n+1$. Consequently, $\det R(\phi)$ must have degree exactly equal to $n+1$ and exactly $n-1$ zero roots. Q.E.D.

Roughly what Theorem 2 says is that in a one consumer separable economy the "eigenvalues" split exactly in the sense that $n-1$ are zero and the $n-1$ zero roots of the backward characteristic polynomial are infinite roots of the forward polynomial and thus are definitely outside the unit circle. The following result is an immediate implication for approximately separable economies.

THEOREM 3. *If a sequence of economies $k = 1, 2, \ldots$ having no zero roots and $D_2 y^k$ non-singular at a steady state converges to an economy with a single separable consumer which satisfies* (R.1)–(R.3), *in the sense that the steady state prices and demand derivatives there converge, then for large enough k the system splits and satisfies the one-good result locally.*

Proof. We consider the forward and backward polynomial $\det R^k(\phi)$ and $\det B^k(\beta)$. By assumption these converge to $R(\phi)$ and $B(\beta)$ and, since $R(\phi) \not\equiv 0$ and $B(\beta) \not\equiv 0$, it can be shown by factorization that the roots converge as well. But then $n-1$ roots of $R^k(\phi)$ go to zero, and, since the roots of $R^k(\phi)$ are the inverses of roots of $B^k(\phi)$, the $n-1$ roots of $B^k(\phi)$ going to zero imply the corresponding $n-1$ roots of $R^k(\phi)$ approach infinity. Consequently, for large enough k, the system splits and, by Kehoe and Levine [7], the one-good result holds. Q.E.D.

Using the topology and methods from Kehoe and Levine [9], we can show that there is an open set of non-degenerate economies ($D_2 y$ non-singular) around the set of degenerate separable economies in which the one-good result holds.

The final step of our argument is to show that the one-good result actually holds for separable economies, and not just their linearized versions.

THEOREM 4. *If* (R.1)–(R.3) *hold and if $D_2 y$ and $D_1 z$ both have rank one in an open neighborhood of the steady state then the one-good result holds locally.*

Proof. Since $D_2 y$ has rank one in a neighborhood of the steady state, the null space of $D_2 y$ is the tangent space to the $n-1$ dimensional manifold of values of p_{t+1} on which the first coordinate function $y^1(p_t, \cdot)$ is constant. By integration $y(p_t, p_{t+1})$ depends on p_{t+1} only through y^1, that is, $y(p_t, p_{t+1}) = \bar{y}(p_t, y^1(p_t, p_{t+1}))$, where $\bar{y}$ is smooth. The equilibrium condition is

$$z(p_{t-1}, p_t) + \bar{y}(p_t, y^1) = 0. \tag{16}$$

Assumption (R.1) implies that $D_1 y + \gamma D_2 z$ is non-singular and thus, for fixed y^1, the set of $q_t = (p_{t-1}, p_t)$ that satisfies (16) is an n-manifold. Also that $D_2 y$ is non-zero implies that as y^1 varies, q_t lies in an $n+1$ manifold Q (compare (11)). Q_L is obviously the tangent space to Q at the steady state. We need to show that the equilibrium condition induces a unique mapping from Q to itself. It would then follow that the linearization on Q is given by part (c) of Theorem 2, which is non-degenerate, and the methods of Kehoe and Levine [9] imply the desired result.

The equilibrium conditions implicitly define a first order correspondence of Q by $z(q_t)+y(q_t^2,q_{t+1}^2)=0$ and $q_{t+1}^1=q_t^2$ where $q_t=(q_t^1,q_t^2)$. Given $q_t\in Q$, $z(q_t)+\bar{y}(q_t^2,y^1)=0$ has a solution $\hat{y}^1(q_t)$ that is locally a smooth function by the implicit function theorem. Consequently, the equilibrium conditions may be written as $y^1(q_t^2,q_{t+1}^2)=\hat{y}^1(q_t)$ and $q_{t+1}^1=q_t^2$, which, since $D_2 y^1$ is not zero, yields an $n-1$ manifold of solutions for q_{t+1} denoted $\bar{Q}(q_t)$ for each q_t. Note that the tangent space to $\bar{Q}(q_t)$ at the steady state value of $q_t=q=(p,\gamma p)$ is just the null space of $D_2 y$.

To solve for q_{t+2} not only must q_{t+1} be an element of $\bar{Q}(q_t)$, but it must be in Q as well. Consequently, we need to show that $Q\cap\bar{Q}(q_t)$ is a singleton. Since Q is $n+1$ dimensional and $\bar{Q}(q_t)$ is $n-1$ dimensional, this is true locally if the manifolds are transverse, and they are transverse locally if they are transverse at the steady state $q_t=q$. Therefore, we need the tangent space to Q, which is Q_L, and to $\bar{Q}(p,\gamma p)$, which is the null space of $D_2 y$, to intersect in a single point. This is of course the point of part (c) of Theorem 2. Q.E.D.

Let us also observe that if there are $n-k$, $n>k>0$ consumers with separable preferences, then $D_2 y$ and $D_1 z$ have rank k and purely notational changes in the argument above establish that there are at most k dimensions of indeterminacy in the nominal case and $k-1$ in the real case.

4. Implications of Small Income Effects

In the complete absence of income effects the matrix of demand derivatives is symmetric so that $(D_1 y+\gamma D_2 z)=(D_1 y+\gamma D_2 z)'$ and $D_1 z=D_2 y'$. Let $R(\phi)$ be the characteristic matrix defined in (3) and let $B(\beta)\equiv\beta^2 R(\beta^{-1})$ be the backward matrix defined in the proof of Theorem 2. Then from (3) and the assumption of symmetry it follows that

$$R'(\phi)=B(\gamma^{-1}\phi). \tag{17}$$

Since R and R' have the same roots and B has the inverse roots, we conclude that, if ϕ is a root of $\det R(\phi)=0$, then so is $\gamma\phi^{-1}$. Since $|\phi|<\sqrt{\gamma}$ if and only if $|\gamma\phi^{-1}|>\sqrt{\gamma}$, this implies in the generic case, where no eigenvalue (except the unit root when $\gamma=1$) exactly equals $\sqrt{\gamma}$, that half the eigenvalues are outside the circle of radius $\sqrt{\gamma}$ and half inside the circle of radius $\sqrt{\gamma}$. We refer to this as *pseudo-splitting*.

Splitting is around the circle of radius γ and pseudo-splitting is around the circle of radius $\sqrt{\gamma}$. At a nominal steady state when $\gamma=1$ pseudo-splitting and splitting are the same; at a real steady state with γ near one we would "generally" expect that the two are the same. In addition when $\gamma<1$ pseudo-splitting implies that at least half the roots are outside the circle of radius γ

and thus that there can be no indeterminacy; when $\gamma > 1$ there can be no instability.

Finally, the continuity of the eigenvalues implies that, if the derivative matrix is approximately symmetric (income effects are "small"), then the implications of pseudo-splitting continue to hold. Thus, the implications of small income effects are similar to the implications of the one-good case. In particular, when $\gamma < 1$ no indeterminacy is possible and when $\gamma = 1$ at most one dimension is possible. Recall that in the model studied by Gale [4] the unique real steady state is autarchic and, when $\gamma < 1$, determinate. Since income effects always vanish at prices that give rise to autarchy, our result can be viewed as a generalization of Gale's. There is, however, no general reason to suppose that with more than one good in a period, more than one consumer in a generation, or more than two periods in a lifetime that a real steady state should be autarchic.

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