Collusion, Randomization and Leadership in Groups

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The Question

- What happens when collusive “Mancurian” groups such as trade-unions, political parties, lobbying organizations and so forth compete in a game?
- The basic setting is one of exogenous groups
- We might expect that: given the play of other groups each group chooses the best strategy for itself
- This does not work as you might hope when the group faces incentive constraints internally
- One of our proposed solutions – leaders with *ex post* evaluation – also has applications to issues of coalition formation traditionally studied in the cooperative game theory literature

Overview

- players are exogenously partitioned into groups within which players are symmetric
- given the play of the other groups there may be several symmetric equilibria for a particular group
- if group can collude they will agree to choose the equilibrium most favorable for its members
- this leads to non-existence
- we augment the model by introducing shadow mixing
- we show how these collusion constrained equilibria arise as the limit of games with perturbed beliefs
- show equivalence to a leadership game with *ex post* evaluation
- builds on models used in mechanism design theory to study collusion in auctions

A Motivating Example

three players

first two players form a collusive group and the third acts independently

theory: given the play of player 3, players 1 and 2 should agree on the incentive compatible pair of (mixed) actions that give them the most utility

each player chooses one of two actions, C or D and payoffs given in bi-matrix form

Payoffs

player 3 plays C payoff matrix for the actions of players 1 and 2 is a symmetric Prisoner's Dilemma game in which player 3 prefers that 1 and 2 cooperate C

	C	D
C	$6, 6, 5$	$0, 8, 5$
D	$8, 0, 5$	$2, 2, 0$

If player 3 plays D the payoff matrix for the actions of players 1 and 2 is a symmetric coordination game in which player 3 prefers that 1 and 2 defect D

	C	D
C	$6, 6, 0$	$4, 4, 0$
D	$4, 4, 0$	$5, 5, 5$

Equilibrium

α^i probability with which player i plays C

set of equilibria for players 1 and 2 given α^3

$\alpha^3 > 1/2$ then D strictly dominant for both player 1 and 2 so they play D,D

$\alpha^3 = 1/2$ two equilibria, both symmetric at C,C and D,D

$\alpha^3 < 1/2$ three equilibria, all symmetric, C,C, D,D and a strictly mixed equilibrium $\alpha^1 = \alpha^2 = (1/3)(1+\alpha^3)/(1-\alpha^3)$

Optimal Collusion

$\alpha^3 > 1/2$ no choice, they have to do D,D (remark: also the unique correlated equilibrium)

$\alpha^3 \leq 1/2$ get 6 at C,C equilibrium and strictly less than 6 at any other correlated within group equilibrium

no ambiguity about the preferences of the group: they unanimously agree in each case as to which is the best equilibrium.

group best response

$\alpha^3 > 1/2$ play D,D

$\alpha^3 \leq 1/2$ play C,C

best response of 3 – never indifferent and always does the wrong thing

group at D,D play D so $\alpha^3 = 0$ at C,C $\alpha^3 = 1$

no equilibrium

Does this make sense?

a small change in the probability of α^3 leads to an abrupt change in the behavior of the group

but how can the group know α^3 so exactly?

rather it makes sense that as the beliefs of a group change the probability with which they play different equilibria varies continuously

$\alpha^3 = 0.499$ versus $\alpha^3 = 0.501$

the theory: player 1 and 2 with probability 1 agree that $\alpha^3 \leq 0.5$ in the former case and in the latter case that $\alpha^3 > 0.5$

perhaps it makes more sense to say that they agree that $\alpha^3 \leq 0.5$with 90% of the time in the former case and mistakenly agree that $\alpha^3 > 0.5$ 10% of the time?

The Cheshire Cat

for the moment suppose that in that limit only the randomization will remain

assume that randomization is possible at the critical point

when $\alpha^3 = 0.5$ and the incentive constraint exactly binds, the equilibrium "assigns" an arbitrary probability to C,C being the equilibrium

if we have 0.5 chance of C,C and D,D then 3 is indifferent and we have an equilibrium

The Exogenous Group Model

players $i = 1 \ldots I$ and groups $k = 1, \ldots K$

actions available for members of group k are A^k a finite set

a fixed assignment of players to groups $k(i)$

all players within a group are symmetric; utility of player i is $u^{k(i)}(a^i, a^{-i})$ and invariant with respect to within group permutations of the labels of other players

$\mathcal{A}^k$ are mixed actions for a member of group k, profiles of play chosen from this set represent the universe in which in-group equilibria reside

each group is assumed to possess a private randomizing device observed only by members of that group that can be used to coordinate group play

restrict to finite subset $A^{kR} \subseteq \mathcal{A}^k$ and consider only in-group equilibria for group k in which all players choose the same action $a^k \in A^{kR}$

Discussion

finiteness simplifies probability distributions over a continuous set

it creates a complication because in-group equilibria may not exist in a finite set

will use approximate equilibrium to take care of that

now write $u^{k(i)}(a^i, a^{k(i)}, \alpha^{-k(i)})$

Collusion

groups collude but must respect incentive constraints

group objective: maximize the common utility that they receive when all are treated equally

Incentive Slack and Shadow Mixing

strictly positive numbers $v^k > 0$ measuring in utility units the violation of incentive constraints that are allowed

gain function

$$G^k(a^k, \alpha^{-k}) = \max_{a^i \in A^k \cup \{a_0^k\}} [u^k(a^i, a^k, \alpha^{-k}) - u^k(a^k, a^k, \alpha^{-k})]$$

degree to which incentive constraint is violated

gain is greater than v^k then the group cannot choose a^k

gain is less than or equal to v^k group may mix with any probability onto a^k if it is at least as good as the best v^k-strict best response

$$U^k(\alpha^{-k}) = \max_{\{a^k | G^k(a^k, \alpha^{-k}) < v^k\}} u^k(a^k, a^k, \alpha^{-k})$$

shadow mixing/best response set

$$B^k(\alpha^{-k}) = \{a^k | G^k(a^k, \alpha^{-k}) \le v^k, u^k(a^k, a^k, \alpha^{-k}) \ge U^k(\alpha^{-k})\}$$

Collusion Constrained Equilibrium

Incentive Compatible Games

If A^{kR} contains a relatively fine grid of mixtures there will be an ϵ-Nash equilibrium with a small value of ϵ

v^k strictly bigger than ϵ the group can find an action that is guaranteed to satisfy the incentive constraints to the required degree

$g^k = \max_{\alpha^{-k}} \min_{a^k \in A^{kR}} G^k(a^k, \alpha^{-k})$: regardless of the behavior of the other groups there is always a g^k approximate equilibrium within the group.

A game is *incentive compatible* if $v^k > g^k$ for all k

existence in incentive compatible games follows from basic continuity properties of the shadow best response correspondence

Random Belief Models and Equilibrium

given the true play $\alpha^{-k}$of the other groups, there is a common belief $\tilde{\alpha}^{-k}$ by group k that is a random function of that true play

An ϵ*-random group belief model* is a density function $f^k(\tilde{\alpha}^{-k}|\alpha^{-k})$ that is a continuous as a function of $\tilde{\alpha}^{-k}, \alpha^{-k}$ and satisfies $\int_{|\tilde{\alpha}^{-k}-\alpha^{-k}|\leq\epsilon} f_\epsilon^k(\tilde{\alpha}^{-k}|\alpha^{-k})d\tilde{\alpha}^{-k} \geq 1-\epsilon$

these can be constructed by standard methods of convolutions; an explicit closed form involving the Dirichlet is given in the paper

$F^k(\alpha^{-k})$ be any probability distribution over v^k- “best best” responses measurable as a function of α^{-k}.

$R^k(a^k|\alpha^{-k}) = \int F^k(\tilde{\alpha}^{-k})[a^k] f^k(\tilde{\alpha}^{-k}|\alpha^{-k})d\tilde{\alpha}^{-k}$.

an ϵ*-random belief equilibrium* as an α_ϵ such that $\alpha_\epsilon^k = R^k(\alpha_\epsilon^{-k})$.

Random Belief vs Collusion Constrained

Theorem: *Fix a family of ϵ-random group belief models, an $F^k(\alpha^{-k})$ and an incentive compatible game. Then for all $\epsilon > 0$ there exist ϵ-random group equilibria. Further, if α_ϵ are ϵ-random belief equilibria and $\lim_{\epsilon \to 0} \alpha_\epsilon = \alpha$ then α is a collusion constrained equilibrium.*

What Difference Do Collusion Constraints Make?

3C

	C	D
C	$6,6,5$	$0,8,5$
D	$8,0,5$	$2,2,0$

3D

	C	D
C	$6,6,0$	$4,4,0$
D	$4,4,0$	$5,5,5$

independent players model: unique Nash equilibrium DDD (5,5,5)

group ignores incentive constraints: unique outcome CCC (6,6,5)

collusion constrained: group shadow mixes 50-50 CC and 3 mixes 50-50 (4.75,4.75,2.5)

- notice that this is worse for everyone than the ordinary Nash equilibrium of the game at 5,5,5
- hence "collusion constrained" - group cannot be stopped from colluding and cannot credibly commit to not doing so
- "Olsonian interest groups"

mechanism designer with safe alternative of (4.9,4.9,4.9)

Leadership Equilibrium

group leaders serve as explicit coordinating devices for groups

we do not want leaders to issue instructions that members would not wish to follow

give them incentives to issue instructions that are incentive compatible by allowing group members “punish” their leader

here v^k has a concrete interpretation as the leader's valence: the higher v^k the more members are ready to give up to follow the leader

leaders give orders that must be followed, but are evaluated *ex post*

A non-cooperative game of leaders

Each group is represented by two virtual players: leader and evaluator with the same underlying preferences as the group members

Each leader has a "big enough" punishment cost P^k . The game goes as follows:

Stage 1: each leader privately chooses an action plan $a^k \in A^{kR}$: conceptually these are orders given to the members who must obey the orders.

Stage 2: the evaluator observes the action plan of the leader of his own group

Stage 3: the evaluator chooses a response a^i

Payoffs: if the evaluator chooses a^k he receives utility $u^k(a^k, a^k, a^{-k}) + v^k$; if he chooses $a^i \neq a^k$ he receives utility $u^k(a^i, a^k, a^{-k})$. If the evaluator chooses $a^i \neq a^k$ the leader is deposed and gets $u^k(a^k, a^k, a^{-k}) - P^k$. Otherwise the leader gets utility $u^k(a^k, a^k, a^{-k})$

Equivalence of Leadership Equilibria

Note that the leader and evaluator do not learn what the other groups did until the game is over.

Theorem: *In an incentive compatible game* α *are sequential equilibrium choices by the leaders if and only if they form a collusion-constrained equilibrium*

Alliances: An Example

the conformists prisoner's dilemma

two symmetric groups with at least three players each

players choose between two actions C, D

if all group action payoffs are

	C	D
C	$1\ ,\ 1$	$-\gamma\ ,\ 1+\gamma$
D	$1+\gamma\ ,\ -\gamma$	$0\ ,\ 0$

individual preferences reflect a desire for conformity: an individual player gets the payoff determined by the common action minus a fixed strictly positive penalty if he fails to choose the group action

any pure choice of action by the group is incentive compatible

basically cooperative game theory

Exogenous Groups

each group has the dominant action of D and the outcome is that this is what both groups do and all players receive 0

but: why should not somebody who can speak to both groups point out the clear benefit to all from forming a single group and make them coordinate on C under his leadership

but: if this happens then why does not a member of, say, group 1 propose that by separating from the common group and playing D?

all members of group 1 would receive $1 + \gamma$ instead of 1

if both groups do this, we are back to 0 and joining the combined group seems attractive again

A Proposal

consider explicitly that there are leaders that recommend actions as before and make utility bids in an effort to form coalitions

group members will choose the best bid

require that bids be credible in the sense that the expected utility group members receive when they choose the best bid should in fact be at least the utility they were promised

suppose there are three leaders:

- two group leaders with preferences inherited from their respective groups, and a common leader who cares about the average utility of all members of both groups
- group leaders send offers only to their own group
- common leader sends offers to both groups

Analysis of the Example

in equilibrium the group leaders always recommend D while the common leader always says C - take this as given

leaders may only bid utility of either 0 or $2(1-\epsilon)$

in case of tie follow the group leader

no pure strategy equilibrium for reasons outlined above

Mixed Equilibrium

group leaders each bids $2(1-\epsilon)$ with probability p

common leader bids $2(1-\epsilon)$ with probability q

accepts an offer of $2(1-\epsilon)$ from the common leader get $-p\gamma + (1-p)\cdot 1$

common leader to be indifferent between the two bids given he will be evaluated *ex post* the expected utility received by the groups should be $1-\epsilon$, since in that case both bids of $2(1-\epsilon)$ and 0 are equally accurate hence the extent to which he may be punished can be determined endogenously to make him indifferent between the two bids

so $p = \epsilon/(1+\gamma)$

similarly $q = (1-\epsilon)/(1+\gamma-\epsilon)$

Qualitative Properties

small ϵ the equilibrium approximately $q = 1/(1+\gamma)$ and $p = 0$

equilibrium probability of cooperation is $q(1-p)^2 \approx q = 1/(1+\gamma)$

γ measures how attractive is defection relative to cooperation

γ small the conflict between the groups is small, the common group forms with high probability and the groups cooperate most of the time

γ large the conflict between the groups is large, the common group forms with low probability and the groups rarely cooperate.

A Model of Endogenous Coalitions

- as sequential equilibrium of a leadership game
- leaders characterized by valences that break ties and groups to whom they can make offers
- evaluators evaluate accuracy of bids along with recommendations of actions
- a subset of the correlated equilibria of the underlying game

Equilibrium in the Conformists Prisoners Dilemma

assume that the grid starts at $\underline{u} < -\gamma$, has gaps of length $0 < d \leq \gamma, 1/2$ and does not contain the points $d/2, 1 + d/2$

there is a strongly symmetric equilibrium in which we denote by $R(u)$ the probability with which a group leader bids less than or equal to u and by $Q(u)$ the probability with which a common leader bids less than or equal to u

which in the limit as $d \to 0$

$R(u) = \frac{u+\gamma}{1+\gamma}$

$Q(u) = e^{(u-1)/\gamma}.$

this has the same basic comparative statics with respect to γ as the simplified example

Group Leaders Who Can Talk

you have to be able to make credible offers – meaning you can only make offers to groups who can punish you

suppose just group leaders, but they can be punished by either group

so they can talk to both groups

let U_0, U_1 be grid points closest to 0 and 1 respectively, consider the bids

$$r^1 = (D, U_0), (C, U_1), \quad r^2 = (C, U_1), (D, U_0)$$

then each group accepts the bid of C from the other group leader

if a group leader tries to outbid the other leader by offering (D, U_1) to his own group then the other group can get at most 0 so he cannot offer the other group more than U_0

that means he loses the other group, and hence the other group will accept their own leader's bid and choose D making a liar of the leader in the eyes of his own group

Conclusion

- the bidding model gives sensible looking equilibria with plausible comparative static properties
- the leadership structure matters
- the mixed equilibrium has strong robustness properties
- the two group leaders equilibrium is overturned if there is a second group leader – who can then outbid the first
- a key issue in applications is to understand which groups leaders with different preferences can talk to
- if a group can choose a leadership structure then choosing a single leader with the same preferences who can be punished by other groups makes sense
- choosing a single leader gives him commitment power, choosing someone who can be punished by other groups enables him to negotiate on your behalf