Superstition and Steady State Learning

Drew Fudenberg and David K. Levine
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Introduction

“If any one bring an accusation against a man, and the accused go to the river and leap into the river, if he sink in the river his accuser shall take possession of his house. But if the river prove that the accused is not guilty, and he escape unhurt, then he who had brought the accusation shall be put to death, while he who leaped into the river shall take possession of the house that had belonged to his accuser.” [2nd law of Hammurabi]
puzzling to modern sensibilities for two reasons

- based on a superstition that we do not believe to be true – we do not believe that the guilty are any more likely to drown than the innocent

- if people can be easily persuaded to hold a superstitious belief, why such an elaborate mechanism? Why not simply assert that those who are guilty will be struck dead by lightning?

from the perspective of the theory of learning in games we ask: which superstitions survive?

- Hammurabi had it exactly right: (our simplified interpretation of) his law uses the greatest amount of superstition consistent with patient rational learning
Overview of the Model

- society consists of overlapping generations of finitely lived players
- indoctrinated into the social norm as children “if you commit a crime you will be struck by lightning”
- enter the world as young adults with prior beliefs that the social norm is true
- being young and relatively patient, having some residual doubt about the truth of what they were taught, and being rational Bayesians, young players optimally decide to commit a few crimes to see what will happen
Example 2.1: The Hammurabi Game

loosely inspired by the law of Hammurabi; player 1 is a suspect; player 2 an accuser; everyone knows the crime has taken place; abstracts from the death penalty

$B$ is the benefit to the accuser of a lie, to the suspect of crime

$P$ is the loss being punished; probability of punishment sufficient to deter crime, $B < pP$
Example 2.2: The Hammurabi Game Without a River
Example 2.3: The Lightning Game

![Diagram of the Lightning Game]

- **Node 1**: Exit
- **Node N**: Crime
- **Transition Probabilities**:
  - $p$: from Node 1 to Node N
  - $1-p$: from Node N to 0
  - $p$: from Node N to B
  - $1-p$: from Node N to B-P
configurations in which there is no crime

Hammurabi game (Nash, but wrong beliefs about off-off path play)

♦ accuser tells the truth because he believes that if he lies he will be punished with probability 1

Hammurabi game without a river (Nash, but not off-path rational)

♦ accuser tells the truth, and is indifferent (ex ante, not ex post)

lightning game (self-confirming, but not Nash)

♦ everyone believes that if they commit a crime they will be punished with probability 1, and that if they exit they will be punished with probability $p$
Simple Games

a simple game

♦ perfect information (each information set is a singleton node)

♦ each player has at most one information set on each path through the tree. (may have more than one information set, but once he has moved, he never gets to move again)

generic condition: no own ties

♦ weaker than no ties – allows the Hammurabi games
The Model

nodes in game tree $x \in X$

pure strategies $s_i \in S_i$, behavior $\pi_i$, mixed $\theta_i$, can be interpreted as fraction of population playing different pure strategies

nodes reached $\bar{X}(\sigma)$ (the “equilibrium path”)

beliefs $\mu_i$ a probability measure over the set of other players’ behavior strategies; when has a continuous density denoted $g_i$

preferences $u_i(s_i, \mu_i)$.
Static Equilibrium Notions

Self-Confirming Equilibrium

Definition 4.1: \( \bar{\theta} \) is a self-confirming equilibrium if for each player \( i \) and for each \( s_i \) with \( \bar{\theta}_i(s_i) > 0 \) there are beliefs \( \mu_i(s_i) \) such that

(a) \( s_i \) is a best response \( \mu_i(s_i) \) and

(b) \( \mu_i(s_i) \) is correct at every \( x \in X(s_i, \bar{\theta}_{-i}) \),

Note also that Nash equilibrium strengthens (b) to hold at all information sets.
In a simple game, node $x$ is one step off the path of $\pi$ if it is an immediate successor of a node that is reached with positive probability under $\pi$.

**Subgame Confirmed Nash Equilibrium**

**Definition 4.2:** Profile $\pi$ is a subgame-confirmed Nash equilibrium if it is a Nash equilibrium and if, in each subgame beginning one step off the path, the restriction of $\pi$ to the subgame is self-confirming in that subgame.
In a simple game with no more than two consecutive moves, self-confirming equilibrium for any player moving second implies optimal play by that player, so subgame-confirmed Nash equilibrium implies subgame perfection.

can fail when there are three consecutive moves.
Example 4.1: The Three Player Centipede Game

unique subgame-perfect equilibrium: all players to pass
(drop, drop, pass) is subgame-confirmed
Rational Steady-State Learning

The Agent’s Decision Problem
“agent” in the role of player $i$ expects to play game $T$ times wishes to maximize

$$\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^{T} \delta^{t-1} u_t$$

$u_t$ realized stage game payoff

agent believes that he faces a fixed time invariant probability distribution of opponents’ strategies, unsure what the true distribution is

Definition 5.1: Beliefs $\mu_i$ are non-doctrinaire if $\mu_i$ is given by a continuous density function $g_i$ strictly positive at interior points.

Note that allow priors can go to zero on the boundary, as is the case for many Dirichlet priors
assume non-doctrinaire prior $g_i^0$

$g_i(\cdot | z)$ posterior starting with prior $g_i$ after $z$ is observed

agents are assumed to play optimally

(dynamic programming problem defined in the paper)

histories are $Y_i$

optimal policy a map $r_i : Y_i \to S_i$ (may be several)
Steady States in an Overlapping generations model

- a continuum population
- doubly infinite sequence of periods
- generations overlap
- $1/T$ players in each generation
- $1/T$ enter to replace the $1/T$ player who leave
- each agent is randomly and independently matched with one agent from each of the other populations

Each population assumed to use a common optimal rule $r_i$

Look for a population steady state in which the fractions of each population playing pure strategies is time invariant
Patient Stability

a sequence of steady states \( \lim_{T \to \infty} \bar{\theta}^T \to \bar{\theta} \) we say that \( \bar{\theta} \) is a \( g^0, \delta \)-stable state.

If \( \bar{\theta}(\delta) \) are \( g^0, \delta \)-stable states and \( \lim_{\delta \to 1} \bar{\theta}(\delta) \to \bar{\theta} \), we say that \( \bar{\theta} \) is a patiently stable state.

**Theorem 5.1:** (Fudenberg and Levine [1993b]) \( g^0, \delta \)-steady states are self-confirming; patiently stable states are Nash.
two profiles $\bar{\theta}, \bar{\theta}'$ are path equivalent if they induce the same distribution over terminal nodes.

a profile is nearly pure if Nature does not randomize on the equilibrium path, and no player except Nature randomizes off the equilibrium path.

our proposed Hammurabi game profile is nearly pure – only Nature randomizes, and only off the equilibrium path.
**Theorem:** *In simple games with no own ties, a subgame-confirmed Nash equilibrium that is nearly pure is path equivalent to a patiently stable state.*

- randomization by players off the equilibrium path – can accomplish this through purification and types

- randomization by Nature on the equilibrium path – in an infinite horizon discounted one-armed bandit problem does the probability of getting stuck on the wrong arm go down at the rate \((1 - \delta)^2\) or faster?

Necessity of subgame-confirmed: affirmative with “independent beliefs” (not in paper)

- without independent beliefs it may be desirable at an off path node to experiment to generate information about an on path node
**Definition:** A profile $\pi$ is ultimately admissible if no weakly dominated strategy (action) is played in an ultimate subgame.

Remark: every subgame confirmed Nash equilibrium is ultimately admissible. In a simple game with no more than two consecutive moves, Nash equilibrium plus ultimate admissability is equivalent to subgame perfection, hence to subgame confirmed Nash equilibrium.

**Theorem:** *Patiently stable states are ultimately admissible Nash equilibria.*

This answers the Hammurabi puzzle: the Hammurabi equilibrium with the river is patiently stable; without the river it is not ultimately admissible; lightning equilibrium even Nash
Games with Length at Most Three

a game has “length at most three” if no path through the tree hits more than three information sets

**Theorem** *In simple games with no own ties, no Nature’s move and length at most three, a subgame-confirmed Nash equilibrium is path equivalent to a patiently stable state.*

because in these games all equilibria are nearly pure

**Lemma:** *In simple games with no own ties, no Nature’s move and length at most three, a subgame-confirmed Nash equilibrium is path equivalent to a subgame-confirmed Nash equilibrium in which players play pure strategies.*

in turn follows from

**Lemma:** *In simple games with no own ties, no Nature’s move and length at most two, every self confirming equilibrium is path equivalent to a public randomization over Nash equilibria.*