Expected Utility Theory

Let $\Omega$ be a probability space

A gamble is a random variable where the quantity represents “money” or “consumption”

Suppose that $x_1$ and $x_2$ are “gambles”

Which gamble is preferred?

Generally: gains are less important than losses
Von Neumann-Morgenstern Preferences

Gambles are compared using a numeric valued utility function

\( u(x) \) is the utility from consuming \( x \)

\( x_1 \) is at least as good (strictly better than) as \( x_2 \)

\( Eu(x_1) \geq (> ) Eu(x_2) \)

risk neutrality: \( u(x) = x \)
Example

\[ u(x) = 10 - \frac{10}{x} \]
Money versus Utility

Money payoffs for player 1

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Utility payoffs for player 1

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>7.5</td>
<td>5</td>
</tr>
</tbody>
</table>
Optimal Choices

If H and T have equal probability is it better to choose U or D?

<table>
<thead>
<tr>
<th></th>
<th>Expected money</th>
<th>Expected utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
<td>6.25</td>
</tr>
</tbody>
</table>

Choose D
Risk Aversion

Would you rather get a gamble $x$ or get the expected value of the gamble $E x$ for sure? Suppose that the gamble is $x^L$ with probability $p$ and $x^H$ with probability $1-p$. 
What happens as $p$ changes?
Risk Loving

Utility

$Eu(x)$

$u(Ex)$

$x^L$  $Ex$  $x^H$

Money
Knee Breakers

risk loving because the loss function is truncated

• you have 1000 and owe a gambling debt of 2000
• double or nothing is a good bet: lose and you still get your knees broken; win and you escape

other applications:
• sporting contests: “the Hail Mary pass”
• the game of banks and regulators
Applications

- Investment: risky portfolio? Stocks or bonds?
- Insurance: auto insurance company charges a premium
  - diversification
  - but hard risk difficult to insure
  - example: industrial decline, everyone should pay, but how to get them to do it?
  - some economists do not really understand competition and see market failure everywhere
  - some “libertarian” non-economists see market failure nowhere
  - in fact market failure is a problem with insurance for large risks
Risk premium

\( y \) a random amount with \( E_y = 0, E_y^2 = 1 \)

relative risk premium \( \rho \)

\[ u(x - \rho x) = E u(x + \sigma y x) \]

\( x - \rho x \) is the certainty equivalent of the gamble

\[ u(x) - \rho xu'(x) = E u(x) + \sigma xu'(x)y + (1/2)\sigma^2 x^2 u''(x)y^2 \]

\[ = u(x) + (1/2)\sigma^2 x^2 u''(x) \]

\[ \rho = -\frac{u''(x)x}{u'(x)} \] coefficient of relative risk aversion
Constant Relative Risk Aversion

\[ u(x) = \frac{x^{1-\rho}}{1 - \rho} \]

also known as “constant elasticity of substitution” or CES

\[ \rho \geq 0 \]

\[ -\frac{u''(x)x}{u'(x)} = \frac{\rho x^{-\rho-1}x}{x^{-\rho}} = \rho \]

\[ \rho = 0 \text{ linear, risk neutral} \]

\[ \rho = 1 \ u(x) = \log(x) \]

useful for empirical work and growth theory, perhaps about two?
**Example**

Logarithmic utility, good approximation in many circumstances

\[ u(x) = \log x \]

endowment: \( x_0 = 100 \)

two investments of 10

stock: 75% gain of 20, 25% no gain

bond: certain gain of 12

<table>
<thead>
<tr>
<th></th>
<th>utility</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>endowment</td>
<td>( \log 100 )</td>
<td>4.605</td>
</tr>
<tr>
<td>stock</td>
<td>(.75 \log 110 + .25 \log 90)</td>
<td>4.650</td>
</tr>
<tr>
<td>bond</td>
<td>( \log 102 )</td>
<td>4.625</td>
</tr>
</tbody>
</table>
What if?

Endowment: \( x_0 = 1000 \)

two investments of 100

stock: 75% gain of 200, 25% no gain

bond: certain gain of 120
Concepts

- expected utility
- **risk aversion**, risk loving, **risk neutral**
- insurance, market failure
- concavity and convexity
- risk premium, certainty equivalent
- **coefficient of relative risk aversion**
Skill

given different investments with different risky returns and a constant relative risk aversion utility function

find which is the superior investment

determine how the answer depends upon risk aversion