PRIVATE OBSERVATION, COMMUNICATION AND COLLUSION

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We examine the possibility of cooperation in a long term relationship, where agents receive diverse imperfect information about each other's actions. "Secret price cutting" in the industrial organization literature is a leading example. In a differentiated product market, a firm cannot directly observe rival firms' secret price cutting, but its own sales can imperfectly indicate what is going on. Since the firms' sales levels are subject to random shocks, they may well end up having diverse expectations: firms with low sales may suspect price cutting while others may not. This causes a serious difficulty in sustaining collusion in such a market. In fact, the characterization of equilibria of this class of games—discounted repeated games where each player receives a different signal—has been an open question, despite the large body of literature on repeated games. The present paper shows that communication is a powerful way of resolving the possible confusion among the players in this class of games. In particular, we construct equilibria where players voluntarily communicate what they have observed and prove folk theorems. Our results thus provide a theoretical support for the conventional wisdom that communication plays an important role in sustaining collusion.

KEYWORDS: Discounted repeated games, folk theorem, imperfect monitoring, privately observed signals, communication, antitrust law, review strategy.

I. INTRODUCTION

THE PRESENT PAPER ANALYZES the role of communication and the possibility of cooperation in a long term relationship, when the actions of the players are imperfectly observed. In particular, we consider the situation where the players receive diverse information about past history and may not share a congruent set of beliefs about what might have happened. In such a situation, we will show that communication is a powerful way of dissolving the possible confusion and coordinating the players' behavior.

"Secret price cutting" is a leading example of the particular situation we analyze in this paper. Consider a small number of firms producing intermediate goods. It is the usual practice in such a market that the effective price of the good is different from the published one, and the former is determined by face-to-face negotiation by the seller and the buyer. This is commonly referred to as "secret price cutting" in the industrial organization literature. As a result,

1 Earlier versions of the present paper have been circulated under different titles, "The Role of Communication in Repeated Games with Imperfect Monitoring" or "Private Observation and Communication in Implicit Collusion." We are thankful to Dilip Abreu, Glenn Ellison, Drew Fudenberg, Eric Maskin, Georg Nódeke, Ariel Rubinstein, and the seminar participants at Summer in Tel Aviv 1992, The 1993 TCEER International Summer Conference, Princeton, Harvard, University of Pennsylvania, Pittsburgh, Columbia, Northwestern, Chicago, and Stockhol. Also detailed comments by Stephen Turnbull, anonymous referees, and an editor greatly improved the paper. The usual disclaimer of course applies.
the firms cannot directly observe others' effective prices. However, each firm can observe its own sales level, which serves as an imperfect signal about other firms' pricing behavior. If sales are low, for example, it may be an indication of other firms' secret price cutting. Or, it may just be the case that market demand is low. An important feature of the market is that the sales level of each firm is private information and cannot be observed by others. This creates a serious difficulty for firms trying to collude for the following reason. To maintain high prices the firms need to punish potential deviators, and this is easiest when they share common beliefs about when a deviation happened and who might be the deviator. In the above situation, however, the firms typically receive different levels of sales and therefore may end up having diverse beliefs about what might have happened.

In fact, the analysis of such a situation is known to be a hard problem in game theory. The situation can be formulated as a repeated game with imperfect monitoring and privately observed signals. Note well that the difficulty is not caused by imperfect monitoring per se, but by the private observability of signals. The celebrated model of collusion by Green and Porter (1984), in which the market price serves as a commonly observable signal, is much more tractable, because the players can easily agree when to punish potential deviators. The study of repeated games with public signals was further extended by Abreu, Pearce, and Stacchetti (1986, 1990), and Fudenberg, Levine, and Maskin (1994) identified sufficient conditions for the folk theorem to hold in such games. In sharp contrast, little is known about repeated games with private signals. This is rather unfortunate because those models have a variety of important economic applications, including such a prominent example as secret price cutting.

There are a limited number of previous contributions on this subject: Radner (1986), a series of works by Lehrer (1989, 1990, 1991, 1992a–c), and Fudenberg and Levine (1991). Those papers, however, share a common weakness. Radner analyzes the no discounting case, in which any act in a finite period, however long it may be, does not affect the total payoff at all. His work extensively uses this property, and cannot readily be extended to the discounted case. In fact, it is known that there is a substantial difference between the discounted and nondiscounted case. Fudenberg and Levine, in contrast, analyze the discounted case, but they assume that the players are only epsilon-rational. When the players are patient, this implies that they do not mind taking suboptimal behavior for a long time, and again it is this rather problematic property that plays a crucial role in their model. Similarly, Lehrer analyzes the no discounting case and/or the discounted epsilon-rational case.

Footnote 3

For the models of publicly observable signals, Radner's positive result (1986) in the no discounting case fails in the discounted model of Radner, Myerson, and Maskin (1986). For the perfect observability case, Fudenberg and Maskin (1986) show the discontinuity between the discounting and no discounting cases.
In the present paper, we analyze perfectly rational players with discounting. Instead of assuming no discounting or irrationality, we introduce communication in the model to overcome the basic difficulty of this subject. We feel that communicating with each other is the most natural way of dissolving confusion among players when they cannot agree on how to maintain collusive behavior. Notice that the conventional wisdom in industrial organization maintains that communication plays an important role in collusion. In fact, certain kinds of communication are per se illegal in the antitrust law. Yet there has been virtually no formal theory to show the role of communication in collusion, and the present paper is hopefully a first step to formulate the conventional wisdom in a general framework.3

In particular, we assume that at the end of each period players can communicate what they privately observed. Communication entails no cost so that it is “cheap talk” rather than “signaling.” We also assume that the players act strategically when they communicate: they can freely provide false information if it suits their best interest. Nevertheless, we will show that we can construct equilibria in which players reveal their private information truthfully, and show that the folk theorem obtains under a set of mild assumptions.

The use of communication in this class of games was first introduced by Matsushima (1990). He analyzed a two-player case and showed that efficiency cannot be achieved by a public perfect equilibrium. In the present paper we show that a folk theorem obtains for a more general class of games and strategies. After the completion of several versions of the present paper, the authors became aware of recent independent contributions by Ben-Porath and Kahneman (1996) and Compte (1998), who also explore the role of communication in repeated games with privately observable signals. Ben-Porath and Kahneman examine the case where each player’s action is perfectly observed by a subset of other players, and prove a folk theorem when each player’s action is observed by at least two players. Compte’s paper is more closely related to ours; it examines the same class of games and proves similar folk theorems with communication. The main differences are (i) we construct equilibria which provide strict incentives for truth telling, and (ii) a different statistical test is employed for the independent signals case. Interested readers are strongly advised to consult his paper.

3 There are many aspects of communication in collusion, and the present paper does not attempt to formulate all of them. For example, one prominent role of communication is to choose which equilibrium to play. While this probably is one of the most important roles of communication in reality, the well-known literature on cheap talk (Farrell (1988), and Matsui (1991)) shows that this aspect of communication is rather hard to formulate in the standard equilibrium analysis. There is also a large body of industrial organization literature on information sharing in static oligopoly problems, where firms receive private signals about the costs or demand and disclose them through communication. As opposed to our model, this literature typically assumes that firms do not provide false information (see Vives (1990) for a survey). We do not need such an assumption in the present paper.
At this juncture let us briefly explain why the analysis of repeated games with privately observed signals has been an open question. In the usual repeated games, players can choose which equilibrium to play depending on the publicly observed signals in each period. This means that after any history, the continuation play is always an equilibrium of the repeated game. This "recursive" structure makes the analysis much easier, and the set of equilibria can be characterized by the dynamic programming technique introduced by Abreu, Pearce, and Stachetti (1986, 1990). On the other hand, when signals are privately observed, continuation plays are no longer equilibria and the recursive structure is destroyed. To see this, consider an equilibrium where players' actions are $a = (a_1, \ldots, a_n)$ in the first period and each player $i$ receives private signal $\omega_i$ according to the distribution $\Pr(\omega_1, \ldots, \omega_n | a)$. If each player $i$ conditions her future strategy on $\omega_i$, the continuation play on the equilibrium path becomes a correlated equilibrium, rather than a Nash equilibrium. More importantly, if player $i$ deviates from $a_i$ to $a'_i$ in the first period, the continuation play is not even a correlated equilibrium. This is because the players have different beliefs about the distribution of the "correlation device" $(\omega_1, \ldots, \omega_n)$ and each other's strategy. Player $i$ knows that the distribution is changed by her deviation $(\Pr(\omega_1, \ldots, \omega_n | a_{-i}, a'_i))$ and may adjust her strategy, while this fact is not known to others. Thus we lose the recursive structure when signals are privately observed, and accordingly there has been no result characterizing the set of equilibria of discounted repeated games with such an information structure.

In the present paper, we overcome this difficulty by introducing communication. Communication generates publicly observable history, and the players can play different equilibria depending on the history of communication. In this way, the recursive structure is recovered, and we are able to use the dynamic programming method developed for the repeated games with publicly observable signals.

The paper is organized as follows. The model is defined in Section 2, and Section 3 summarizes the basic technique from the previous work and provides an overview of the basic ideas. The reader who is not interested in the technical details is advised to read Section 3.2 to understand the basic theoretical constructions of our results and the relationship to the existing literature. Section 4 considers the case where the players seriously communicate every period and identifies the conditions for the folk theorem. In Section 5 we show that efficiency can sometimes be improved by infrequent communication, together with the statistical testing addressed by Radner (1986) and Matsushima (1995). Concluding remarks are given in the last section.

2. THE MODEL

The stage game $G$ is defined as follows. Each player $i \in N = \{1, \ldots, n\}$ simultaneously chooses an action $a_i$, and after choosing it she observes her own private signal $\omega_i$ which is not observed by the opponents. Let $A_i$ be the finite set of actions for player $i$, and let $\Omega_i$ be the finite set of possible private signals for
player $i$. We denote $\prod_{i \in N} A_i = A$ and $\prod_{i \in N} \Omega_i = \Omega$, and their generic elements are denoted $a$ and $\omega$ respectively. Similar notations will be employed for product sets and vectors in what follows. The probability distribution of private signal profile $\omega$ conditional on action profile $a$ is denoted $p(\omega | a)$. We assume this distribution has full support, that is, for each $a \in A$ and each $\omega \in \Omega$, $p(\omega | a) > 0$.

Player $i$'s instantaneous payoff $u_i(a_i, \omega_i)$ is determined by her own action and her own private signal only, i.e., it is independent of the opponents' actions

$$a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

and their private signals

$$\omega_{-i} = (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n).$$

This formulation makes sure that the realized payoff $u_i$ reveals no more information than $a_i$ and $\omega_i$ do. In the secret price cutting application, $u_i$ is the profit of firm $i$, which depends on its price $a_i$ and the quantity sold $\omega_i$. Player $i$'s expected payoff when players choose action profile $a \in A$ is

$$g_i(a) = \sum_{\omega \in \Omega} u_i(a_i, \omega_i)p(\omega | a).$$

Let $\sigma_i$ be player $i$'s mixed action, $\Delta_i$ the set of player $i$'s mixed actions. Let $g_i(\sigma_i)$ and $p(\omega | a)$ be player $i$'s expected payoff and the probability of $\omega$ respectively when players conform to mixed action profile $\sigma \in \Delta$.

We will allow players to communicate with each other. After choosing actions and observing private signals, the players simultaneously and publicly announce messages $(m_1, \ldots, m_n)$. Let $M_i$ be the finite set of player $i$'s possible messages, which will be specified in what follows. Define $\Delta(M_i)$ to be the set of probability distributions over $M_i$.

The infinitely repeated game with discounting associated with the stage game $G$ is denoted by $\Gamma(G, \delta)$, where $\delta \in (0, 1)$ is the discount factor. A strategy for player $i$ is defined by $s_i = (\sigma_i, \eta_i)$, where $\sigma_i$ and $\eta_i$ specify action and message respectively. In particular,

$$\sigma_i = (\sigma_i(t))_{t=1}^\infty,$$

$$\eta_i = (\eta_i(t))_{t=1}^\infty,$$

$$\sigma_i(t): A_{i-1}^{i-1} \times \Omega_i^{i-1} \times M^{i-1} \rightarrow \Delta_i,$$

and

$$\eta_i(t): A_i \times \Omega_i \times M^{i-1} \rightarrow \Delta(M_i).$$

We denote $(m(1), \ldots, m(t))$ by $m^t$. Similar definitions apply for $a_i^t$, $\omega_i^t$, etc., and we define private history by $(a_i^t, \omega_i^t) = h_i^t$ and (joint) history by $(m^t, h_i^t, \ldots, h_n^t) = h^t$. Player $i$'s expected average payoff when players conform to strategy profile $s = (s_1, \ldots, s_n)$ is

$$u_i(s, \delta) = (1 - \delta)E\left[ \sum_{t=1}^{\infty} u_i(a_i(t), \omega_i(t)) \delta^{t-1} | s \right],$$
where $E[\cdot | s]$ is the expectation with respect to the probability measure on histories induced by strategy profile $s$, and $a_i(t)$ and $\omega_i(t)$ are the action and the private signal for player $i$ realized in period $t$. The solution concept is sequential equilibrium. The set of sequential equilibrium average payoffs is denoted $V(G, \delta)$. Define $V(G) = \lim_{\delta \to 1} V(G, \delta)$.

3. CHARACTERIZATION AND OVERVIEW

To analyze the equilibria in the discounted repeated game $\Gamma(G, \delta)$, we employ the method developed by Fudenberg and Levine (1994).\footnote{Fudenberg and Levine consider a very general case where there are two types of players, long-run and short-run. The time horizon for a long-run player is infinite, while a short-run player lives only one period and will be replaced by a newcomer. Note that their model does include the usual repeated games played only by long-run players as a special case.} Instead of directly solving the repeated game, this method first considers simple contract problems associated with the stage game. Then, the solutions to those contract problems are utilized to construct the set of equilibrium payoffs of the repeated game. In Subsection 3.1 we present a version of their method modified to fit our framework with private signals and communication. Then, for the reader's convenience, we will provide an intuitive explanation about why this method works. Lastly, a nontechnical overview of our construction of folk theorems will be given in Subsection 3.2.

3.1. Characterization of Equilibrium Payoff Set

Now let $G^T(\delta)$ denote the $T$-time repeated version of the stage game $G$ with discount factor $\delta$, where the realized payoff for player $i$ is given by the average discounted sum of the stage payoff,

$$\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^{T} u_i(a_i(t), \omega_i(t)) \delta^{t-1}.$$

Then modify this by attaching a sidepayment contract $x^T = (x^T_i)_{i \in N}$, where $x^T_i : M^T \to R$. Here, $x^T_i(m^T)$ is the sidepayment (utility transfer) to player $i$, paid at the end of the game, when the history of message profiles $m^T$ is realized. This means that player $i$'s realized payoff is

$$\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^{T} u_i(a_i(t), \omega_i(t)) \delta^{t-1} + x^T_i(m^T),$$

which defines finitely repeated game with sidepayments $(G^T(\delta), x^T)$. Roughly speaking, this corresponds to the dynamic programming decomposition of the repeated game into the first $T$ periods and the continuation play. A detailed discussion will be given shortly. Player $i$'s strategy in this game, $s^T = (\sigma^T_i, \eta^T_i)$, is defined in the same way as in the previous section. Player $i$'s expected payoff
under strategy profile $s^T$ is

$$v_i^T(s^T, x_i^T, \delta) = E\left[\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^{T} u_i(a_i(t), \omega(t))\delta^{t-1} + x_i^T(m^T)|s^T\right],$$

where $E[\cdot|s^T]$ is the expectation with respect to the probability measure on histories induced by strategy profile $s^T$.

A strategy profile in any extensive form game is said to have no observable deviation if, for each player, any of her unreached information sets can be reached by and only by her own deviations. Given this definition and a given discount factor $\delta^0 \in (0, 1)$, we shall confine our attention to strategy profiles $s^T$ which satisfy the following conditions.

[(SA) (STATIONAL ACTION): On the equilibrium path $s^T$ specifies the same (possibly mixed) action profile $\alpha$ for each period.]

[(NOD): $s^T$ has no observable deviation, and conditional on any $(a_i^T, \omega_i^T, m^{T-1})$, the distribution of $m_i(T)$ has full support.]

[(\delta^0-UIC) (UNIFORM INCENTIVE COMPATIBILITY): $s^T$ is a Nash equilibrium in $(G^T(\delta), x^T)$ for all $\delta \in [\delta^0, 1)$.

Note that condition (SA) implies that the expected average payoff is independent of the discount factor:

$$v_i(s^T, x_i^T, \delta) = g_i(\alpha) + E[x_i^T(m^T)|s^T].$$

Now, for every welfare weight $\lambda \in \mathbb{R}^n \setminus \{0\}$, we introduce the following problem to design the optimal sidepayment contract $x^T$:

**PROBLEM ($T, \lambda, \delta^0$):**

$$\max_{(\alpha, s^T, x^T)} \sum_{i \in N} \lambda_i \left[ g_i(\alpha) + E[x_i^T(m^T)|s^T] \right]$$

subject to (SA), (NOD), ($\delta^0$-UIC), and

(B) $\sum_{i \in N} \lambda_i x_i^T(m^T) \leq 0$ for all $m^T \in M^T$.

Let $k(T, \lambda, \delta^0)$ denote the optimal value of this problem. Define

$$D(T, \lambda, \delta^0) = \{ v \in \mathbb{R}^n | \lambda v \leq k(T, \lambda, \delta^0) \}, \quad \text{and}$$

$$Q(T, \delta^0) = \bigcap_{\lambda \neq 0} D(T, \lambda, \delta^0).$$

The above contract problem is useful in studying the equilibrium payoff set of the infinitely repeated game, thanks to the following fact.
PROPOSITION 1: $Q(T, \delta^0)$ is a subset of $V(G)$ if the dimension of $Q(T, \delta^0)$ is equal to the number of players.

This is a modification of Fudenberg and Levine's result (1994), and the proof is given in the Appendix. An intuitive explanation will be given shortly.

Now define

$$Q(T) = \lim_{\delta^0 \to 1} Q(T, \delta^0).$$

Since Proposition 1 holds for all $\delta^0 \in (0, 1)$, $Q(T)$ is a subset of $V(G)$ if it is full-dimensional. We finally define

$$Q = \lim_{T \to \infty} Q(T).$$

Clearly we have the following.

COROLLARY 1: $Q$ is a subset of $V(G)$ if the dimension of $Q$ is equal to the number of players.

In what follows, we will show that, under certain conditions, $Q$ is equal to the feasible and individually rational payoff set to prove folk theorems.

For each $T = 1, 2, \ldots$, set $Q(T)$ represents the collection of limit equilibrium payoffs (as $\delta \to 1$) in a different class of strategies in the infinitely repeated game. If $T = 30$ (a month), for example, $Q(T)$ admits the situation where each player utilizes private information (her action and signal) only within a given month. On any given day in the first month (say, January), each player chooses her action based on her past actions and private signals, as well as the history of publicly exchanged messages. In February, however, the players abandon what they privately observed in January, and condition their actions on (i) publicly exchanged messages in January and February and (ii) private information in February. Hence the players periodically abandon all private information every $T$ periods (i.e., at the beginning of each month), but they (potentially) utilize the whole history of publicly exchanged messages. As we will explain in more detail, this linkage to other months is captured by the term $x^T(m^T)$ in Problem $(T, \lambda, \delta^0)$, representing the effect of the message $m^T$ on continuation payoffs.

Formally, strategy $s_t$ is $T$-public if for all $k = 0, 1, 2, \ldots$ and all $t > kT$, $s_t(t)$ is independent of private history up to $kT(h^{kT})$. A Nash equilibrium is $T$-public perfect if it is $T$-public and after any history $h^{kT}$ ($k = 1, 2, \ldots$) it specifies a Nash equilibrium. Then, the proof of Proposition 1 shows that $Q(T)$ represents a collection of limit equilibrium payoffs (as $\delta \to 1$) of $T$-public perfect equilibria. As $T$ increases, more dependence on private information is allowed, so we have $Q(T) \subset Q(T')$ for $T' = LT$, where $L$ is a positive integer.

Now we provide a brief intuition for the Fudenberg-Levine algorithm (Proposition 1), which connects the optimal contract problem $(T, \lambda, \delta^0)$ with the repeated game equilibria. Let us consider the case where the players abandon
private information every $T$ periods (i.e., $T$-public perfect equilibria). When player $i$ receives an average payoff of $v_i$, it can be decomposed into the payoff in the first $T$ periods and the continuation payoff after that (denoted $w_i$):

$$
v_i = (1 - \delta^T)E\left[\frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^{T-1} u_i(a_i(t), \omega_i(t)) \delta^t s^t \right] + \delta^T E\left[w_i(m^T)\right] s^T \]

$$

$$
= (1 - \delta^T)\pi_i + \delta^T E\left[w_i(m^T)\right] s^T \]

$$

where $s^T$ and $m^T$ respectively represent the strategies and the messages sent in the first $T$ periods, and $\pi_i$ is the average payoff in the first $T$ periods (it is equal to $g_i(\alpha)$ on the equilibrium path when $s^T$ satisfies $\text{(SA)}$). If we rewrite $w_i(m^T)$ as $v_i + x_j^T(m^T)(1 - \delta^T)/\delta^T$, we have the key equation $v_i = \pi_i + E[x_j^T(s^T)]$. Note that $x_j^T(1 + \delta + \cdots + \delta^{T-1})$ represents the total (as opposed to average) variation of the continuation payoff around $v_i$ (i.e., $(w_i - v_i)\delta^T/(1 - \delta)$).

Let $W$ in Figure 1 denote the set of average continuation payoffs. In a $T$-public perfect equilibrium, continuation payoff profile $w(m^T)$ is always chosen from this set $W$. Note that the players' actions in the first $T$ periods influence the choice of the continuation payoffs through the exchanged messages. Hence in the construction of an equilibrium, we must choose the continuation payoffs judiciously so that (i) the players are willing to send informative messages and (ii) deviants in the first $T$ periods are punished. Furthermore, if we are interested in the best repeated game equilibrium in a particular direction $\lambda$, we must make the continuation equilibria as close as possible to the boundary of $W$ in the direction of $\lambda$ (see Figure 1(a)). Now notice that the points in set $W$ in Figure 1(a) are average (per period) payoffs. When the discount factor $\delta$ is close to 1, a small variation of $w$ creates a huge change in total payoffs. This means that we can then choose $w(m^T)$ from the small neighborhood in Figure 1(a), and still be able to provide sufficient incentives. Hence, if the discount factor is arbitrarily close to unity, we can regard the total variation of continuation

![Figure 1](image-url)
payoffs being effectively chosen from the half space $H$ defined by the normal vector $\lambda$ (Figure 1(b)). This is the basic idea of the algorithm; the "sidepayment" $\mathbf{x}^T$ in the contract problem $(T, \lambda, \delta^{T-1})$, when multiplied by $(1 + \delta + \cdots + \delta^{T-1})$, corresponds to the variation (in terms of total present value) of continuation payoffs, and the "budget constraint" $(B)$ indicates that they are chosen from a tangent half space of the equilibrium payoff set as in Figure 1(b).

3.2. A Nontechnical Overview

With the technique developed above, we will provide several versions of the folk theorem in the following sections. Before going into the technical details, let us sketch the basic ideas. The main idea is based on the following result (Williams and Radner (1995), Matsushima (1989), and Fudenberg, Levine, and Maskin (1994)). Roughly speaking, efficiency under publicly observable signals can be achieved if players can be punished by "transfers." That is, if the information structure allows us to tell which player is suspect, we can transfer the suspect player's future payoff to the other players. This can provide the right incentives without causing welfare loss, compared to the case where all players are punished simultaneously. If the signals are privately observed, however, we must induce each player to reveal her signal truthfully, and this imposes certain restrictions on the form of feasible payoff transfers. The easiest way to solicit truthful information is to make each player's future payoff independent of what she communicates. If this is the case, she is just indifferent as to what she says, and truthful revelation becomes a (weak) best response.

This can be done if there are at least three players and the information structure can distinguish different players' deviations. A player's private information can be used to determine when and how to transfer payoffs among other players. This possibility is explored in Section 4. Section 4 also examines the possibility of providing strict incentives to tell the truth. We show that when private signals are correlated, there is a way to check if each player is telling the truth. All of this is done by assuming serious communication every period: in other words, we look at 1-public perfect equilibria and we utilize the above characterization of $Q(1)$ accordingly.

If there are two players, or if the information structure fails to distinguish different players' deviations, the above idea cannot be utilized. Accordingly, we will explore in Section 5 the possibility of a folk theorem by means of $T$-public perfect equilibrium with $T > 1$. That is, if the players (seriously) communicate only every $T$ periods and their actions can partly depend on their private information, there is a possibility of getting better outcomes. Such a possibility has been demonstrated by Abreu, Milgrom, and Pearce (1991), who showed that the delay of the release of public information can enhance efficiency in repeated

\footnote{For this to be true, $W$ must be a set with smooth boundary. The Fudenberg and Levine algorithm exploits the fact that any smooth subset of $Q$ (which is a polyhedron, thus not smooth) can be a subset of the equilibrium payoff set if the discount factor is close enough to one.}
he normal "sidepay- 

\[ (1 + \delta + \text{value}) \] of 

\text{t} they are 

\[ \text{figure 1(b).} \]

games. With privately observed signals, it turns out that their method is not 
directly applicable, and we instead use a new technique. Note also that the delay 
of information release is exogenously imposed in Abreu et al., while it is 
endogenously derived in our model. We will show that, as the discount factor \( \delta \) 
tends to 1 and the interval of serious communication \( T \) tends to infinity, the folk 
theorem holds under some conditions.

4. FOLK THEOREMS WITH FREQUENT COMMUNICATION

In this section we present folk theorems when the players seriously communicate 
in each period (i.e. when they play a 1-public perfect equilibrium). This is possible 
when the number of players is more than two \((n > 2)\) and the information structure can distinguish different players' deviations. We start with a set of 
sufficient conditions for the folk theorem.

4.1. Sufficient Conditions

Let \( \Omega_i = \Pi_{k^{*}i} \Omega_k \) and \( \Omega_{-ij} = \Pi_{k^{*}i,j} \Omega_k \), and define vectors 
\[
p_{-i}(a) \equiv (p_{-i}(\omega_{-i} | a))_{\omega_{-i} \in \Omega_{-i}} \quad \text{and} \\
p_{-ij}(a) \equiv (p_{-ij}(\omega_{-ij} | a))_{\omega_{-ij} \in \Omega_{-ij}},
\]

where \( p_{-i}(\omega_{-i} | a) \) and \( p_{-ij}(\omega_{-ij} | a) \) are marginal distributions of \( p(\omega | a) \). 
Vectors \( p_{-i}(a) \) and \( p_{-ij}(a) \) represent the distributions of signals, given action profile \( a \), observed by player \( i \)'s opponents and by player \( i \)'s and \( j \)'s opponents respectively. For brevity, we call the set of players other than \( i \) and \( j \) 
"\( ij \)-opponents." Conditional distributions given a mixed action profile \( a \in \Delta \) are 
denoted \( p_{-i}(a) \) and \( p_{-ij}(a) \). Note that \( p_{-ij} \) is well defined only when there are 
more than two players, which will be assumed throughout Section 4.

Let \( \mu^i \) be the minimax profile (in mixed strategies) for player \( i \):

\[
\mu^i_{-i} \in \arg \min_{\alpha_{i}, \in \Delta_{-i}} \left( \max_{a_{i}, \in A_{i}} g_{i}(a_{i}, \alpha^i_{-i}) \right), \\
\mu^i_{i} \in \arg \max_{a_{i}, \in A_{i}} g_{i}(a_{i}, \mu^i_{-i}).
\]

The first assumption we employ is the following.

(A1): For all \( i \) and \( j \neq i \), if there is a mixed strategy \( \alpha_{j} \in \Delta_{j} \) such that 
\( p_{-j}(\mu^i_{-i}, \alpha_{j}) = p_{-j}(\mu^i_{-i}, \alpha_{j}) \), then \( g_{j}(\mu^j_{-j}, \alpha_{j}) \geq g_{j}(\mu^j_{-j}, \alpha_{j}) \).

This assumption will be utilized to provide proper incentives for a player to 
punish (minimax) another. Assumption (A1) requires that if player \( j \) has a 
perfectly undetectable deviation \( (\alpha_{j}) \) at the minimax point for player \( i \) \( (\mu^i_{-i}) \), \( j \) 
has no incentive to take it (i.e. it reduces \( j \)'s stage payoff). Note well that the
"perfect undetectability" of the deviation $\alpha_i$ in the above sentence has a very strong meaning. It requires that both $\mu'_j$ and $\alpha_j$ produce exactly the same distribution of the signals observed by $j$'s opponents ($\omega_{-j}$). This should not be confused with the undetectability in a weaker sense. Under our full support assumption for the signals, any outcome $\omega_{-i}$ can always be realized with a positive probability, irrespective of the action taken. So it is a fortiori impossible to determine whether player $j$ is actually punishing $i$ (using $\mu'_j$) or not ($\alpha_j$) for sure. However, if $\mu'_j$ and $\alpha_j$ produce different distributions of $\omega_{-j}$, the expected reward (future payoffs) for player $j$, which is a function of $\omega_{-j}$, can change when $j$ switches from $\mu'_j$ to $\alpha_j$. Hence, if the premise of (A1), $p_{-j}(\mu') = p_{-j}(\mu'_{-j}, \alpha_j)$, is violated, there is a possibility to provide an incentive for player $j$ not to take action $\alpha_j$. In what follows we will show that this is indeed the case. With this observation in mind, we will say that a certain deviation is statistically detectable if it changes the distribution of the signals.

Now define, for each pair $i \neq j$ and each profile $a \in A$,

$$Q_{ij}(a) = \{ p_{-i,j}(a_{-i}, a'_j) \mid a'_j \in A_j \setminus \{a_j\} \}.$$  

This is a collection of distributions of $ij$-opponents’ signals, generated by player $i$’s deviations from the profile $a$. Let $Ex(A)$ be the set of strategy profiles which provide the extreme points of the stage payoff set. Given any set $X$, we denote its convex hull by $\text{co}(X)$.

(A2): For each pair $i \neq j$ and each $a \in Ex(A),

$$p_{-i,j}(a) \in \text{co}(Q_{ij}(a) \cup Q_{ji}(a)).$$

(A2) says that if either player $i$ or $j$ (but not both) deviates with certain probabilities, the other players can statistically detect it. In particular implies the following:

(A2′): For each pair $i \neq j$ and $a \in Ex(A), p_{-i,j}(a) \in \text{co}(Q_{ij}(a)).$
This says that player $i$'s mixed strategy deviations are statistically detected by $ij$-opponents. This in turn trivially implies the following:

$$(A2^*): \text{For each } i \text{ and } a \in \text{Ex}(A),$$

$$p_{-i}(a) \notin \text{co}(\{p_{-i}(a_{-i}, a'_i) \mid a'_i \in A_i \setminus \{a_i\}\}).$$

That is, any mixed strategy deviation by player $i$ is statistically detected by her opponents.

$$(A3): \text{For each pair } i \neq j \text{ and each } a \in \text{Ex}(A),$$

$$\text{co}(Q_{ij}(a) \cup \{p_{-ij}(a)\}) \cap \text{co}(Q_{ji}(a) \cup \{p_{-ji}(a)\}) = \{p_{-ij}(a)\}.$$ 

This says that $ij$-opponents can statistically discriminate player $i$'s (possibly mixed) deviations from player $j$'s, because they always create different distributions of $\omega_{-ij}$.

Let us now examine when the information conditions $(A1)$–$(A3)$ are likely to be satisfied. They are somewhat weaker versions of the “individual full rank” and “pairwise full rank” conditions by Fudenberg, Levine, and Maskin (1994). Assumption $(A1)$ is vacuously satisfied when different pure strategy deviations by player $j$ create linearly independent distributions of the signals $\omega_{-j}$, because $p_{-j}(\mu^j) = p_{-j}(\mu^j_j, \alpha_j)$ if and only if $\alpha_j = \mu^j_j$ in such a case. This linear independence requirement corresponds to the individual full rank condition by Fudenberg-Levine-Maskin. A necessary condition for this is that the number of

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6 Our conditions are weaker because they are stated in terms of a convex combination of distributions, while Fudenberg, Levine, and Maskin consider a linear combination. Note that our conditions can also be used in their model to prove the folk theorem with public signals.
possible realizations of $\omega_{-ij}$ be larger than $j$'s pure actions, that is, $L^{n-1} \geq K$ if we denote $|A_j| = K$, $|\Omega_i| = L$ \( \forall i \) (recall that $n$ is the number of players). Similarly, (A2) and (A3) are satisfied when the original signal distribution and the distributions created by player $i$'s and $j$'s unilateral deviations (that is, $p_{-ij}(a)$ and the elements in $Q_{ij}(a)$ and $Q_{ji}(a)$) are linearly independent (the pairwise full rank condition by Fudenberg-Levine-Maskin). A necessary condition for this is

(4.1) \[ L^{n-2} \geq 2K - 1. \]

The left-hand side is the number of possible realizations of $\omega_{-ij}$, and the right-hand side is the number of the relevant distributions, the original distribution and those created by $K-1$ deviations by each player $i$ or $j$. Observe that (4.1) is stronger than the former condition $L^{n-1} \geq K$.

For a generic choice of signal distributions $\rho(\omega | a)$, on the other hand, the individual and pairwise full rank conditions are satisfied under (4.1). In the secret price cutting application, $K$ is the number of possible prices a firm can charge, and $L$ is the number of possible sales levels. Although one might expect that $L$ is smaller than $K$ in this case, the left-hand side of (4.1) grows very quickly (exponentially) as the number of firms ($n$) increases. For example, even if $L = 100$ and $K$ is one million, (4.1) is satisfied when there are six firms. So we can conclude that assumptions (A1)–(A3) are generically satisfied in the secret price cutting application if the number of the firms is not too small.

Next let us turn to see when our assumptions might fail. Consider a completely symmetric case when a $1$ price cut by firms 1 and 2 has exactly the same impact on the remaining firms' sales. In such a case, (A3) is violated. However, even the slightest asymmetry, caused by locations or product differentiation, restores our generic assumptions, as long as (4.1) is satisfied. On the other hand, (4.1) itself might be violated if the signal space is not rich enough and the number of firms is very small. In such a case, a folk theorem cannot be obtained by the class of equilibria considered in the present section. In Section 5, however, we will show the possibility of improving efficiency by infrequent communication.

Now let us explain how those assumptions are used to prove a folk theorem. The following lemma provides the key implications of (A2) and (A3).

**Lemma 1:** Under (A2) and (A3), for each pair $i \neq j$, $\lambda_i \neq 0$, $\lambda_j \neq 0$, and $a \in Ex(A)$, and for any positive number $d$, we can construct payment schemes $x_i, x_j$ such that

(i) $\lambda_i x_i(\omega_{-ij}) + \lambda_j x_j(\omega_{-ij}) = 0 \; \forall \omega_{-ij}$,

(ii) $E[x_h | a] - E[x_h | a_{-h}, a'_h] \geq d \; \forall a_h, h = i, j$.

Such payment schemes can be constructed from the separating hyperplanes depicted in Figures 2 and 3 (see Appendix for the details). Let us now briefly explain the meaning of the above lemma and how to go about proving the folk
theorem (Theorem 1 below). In the repeated game equilibrium, what if-opponents say \( (m_{-ij} = \omega_{-ij}) \) determines the future payoffs of players \( i \) and \( j \), which are constructed from the transfer rules described in Lemma 1. Lemma 1 states that we can find a transfer rule for each pair of players \( (x_i, x_j) \) which provides a penalty for any deviation (condition (ii)) and entails no social welfare loss (condition (i)). Once we define such a transfer rule for each pair, we will "pack together" all those payment schemes. Note that for each player \( i \), \((n - 1)\) payment schemes \( x_1^i, \ldots, x_i^n \) are defined (as there are \( n - 1 \) pairs to which player \( i \) belongs). Then we take the summation of them, \( X_i = x_1^i + \ldots + x_i^n \). If we choose the number \( d \) in Lemma 1 sufficiently large, this patchwork generates payment schemes \( (X_1, \ldots, X_n) \) which satisfy:

(i) \( \lambda_i X_i + \ldots + \lambda_n X_n = 0 \) (no welfare loss);
(ii) \( E[X_k | a] - E[X_h | a, a'] \geq g_h(a, a') - g_k(a), \forall a_k \forall h \) (providing correct incentives); and
(iii) \( X_h \) is independent of \( m_h \) (so that each player \( h \) has a (weak) incentive to tell the truth).

Those payment schemes correspond to the variations of equilibrium continuation payoffs in the repeated game. This way, we can construct an efficient equilibrium, as is explained in Subsection 3.2. This is a rough sketch of the proof of the folk theorem stated below (Theorem 1).

Before stating the folk theorem, let us introduce some notation. Let \( \nu_i^* = g_i(\mu') \) be the minimax value for player \( i \) and define the feasible and individually rational payoff set by

\[ V^* = \{ v \in \text{co}(g(A)) | v \geq \nu^* \} \]

Now we are ready to present a folk theorem result. The formal proof is left to the Appendix.

**Theorem 1:** Suppose that there are more than two players \((n > 2)\) and the information structure satisfies condition (A1), (A2), and (A3). Also suppose that the dimension of \( V^* \) is equal to the number of players. Then, any interior point in \( V^* \) can be achieved as a sequential equilibrium average payoff profile of the repeated game with communication, if the discount factor \( \delta \) is close enough to 1.

### 4.2. Strict Incentives for Truth-Telling

So far, we have only required the weak sense of truthful revelation. To show Theorem 1 we constructed equilibria where each player is indifferent about what to communicate. We will show below that when private signals are mutually correlated, we can provide the players with strict incentives to tell the

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7 What follows is the intuitive explanation for the case where each element of vector \( \lambda \) is strictly positive. Other cases are treated similarly. See the proof of Theorem 1 for the details.

8 Those conditions are mainly imposed on the information structure. We can weaken those conditions by imposing assumptions both on the information structure and the payoffs functions. Such a possibility has been explored in the discussion paper version of the paper.
truth, by adding a rather mild informational assumption. For simplicity, we first consider minimax points defined with respect to pure strategies. Let $b^i \in A$ be the minimax point for player $i$ with respect to pure strategies, and let $v_i^{**} = g_i(b^i)$. Accordingly, define the feasible and individually rational payoff set by

$$V^{**} = \{ v \in \text{co}(g(A)) \mid v \geq v^{**} \}.$$

For $\lambda$ such that $\lambda_i < 0$ for some $i$ and $\lambda_j = 0$ for all $j \neq i$, let $a(\lambda) = b^i$, and otherwise $a(\lambda) \in \arg \max_{a \in A} \lambda g(a)$. We will employ the following condition to provide strict incentives for truth-telling:

(A4): For every $a(\lambda)$, $\lambda \neq 0$, every $i$, every $\omega_i, \omega'_i \in \Omega$, with $\omega_i \neq \omega'_i$,

$$\tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i) \neq \tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega'_i)$$

for some $\omega_{-i} \in \Omega_{-i}$,

where $\tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i)$ is the conditional distribution of $\omega_{-i}$ given $a(\lambda)$ and $\omega_i$.

This assumption says that the private signals are correlated in such a way that each realization of a player's private signal induces a different conditional distribution of the opponents' signals.

We define $f_i : \Omega \rightarrow \mathbb{R}$ by

$$f_i(\omega) = 2\tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i) - \sum_{\omega'_{-i} \in \Omega_{-i}} \tilde{p}_{-i}(\omega'_{-i} \mid a(\lambda), \omega_i)^2.$$

As we will see, adding $f_i$ to player $i$'s continuation payoff provides a strict incentive for truth-telling. To understand this construction, consider the following "game." Player $i$ is asked to announce a function $q : \Omega_{-i} \rightarrow \mathbb{R}$, and will get payoff

$$2q_i(\omega_{-i}) - \sum_{\omega'_{-i} \in \Omega_{-i}} q(\omega'_{-i})^2.$$

Given player $i$'s information her expected payoff is

$$\sum_{\omega_{-i} \in \Omega_{-i}} \left( 2q(\omega_{-i}) - \sum_{\omega'_{-i} \in \Omega_{-i}} q(\omega'_{-i})^2 \right) \tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i)$$

and the first order conditions for the optimal announcement $q$ is

$$2\tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i) - 2q(\omega_{-i}) = 0 \quad \text{for all } \omega_{-i} \in \Omega_{-i}.$$

Since the second order conditions are satisfied (i.e., $-2 < 0$), announcing her conditional distribution is the unique optimum in this "game;" $q(\omega_{-i}) = \tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i)$ for all $\omega \in \Omega$. Since there is one-to-one correspondence between her private signal and the conditional distribution under (A4), this implies that

$$\sum_{\omega_{-i} \in \Omega_{-i}} f_i(\omega) \tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i)$$

$$> \sum_{\omega_{-i} \in \Omega_{-i}} f_i(\omega_{-i}, \omega'_i) \tilde{p}_{-i}(\omega_{-i} \mid a(\lambda), \omega_i)$$
for all \( \omega_i \neq \omega_j \). That is, player \( i \) has a strict incentive for truth-telling if she receives payment \( f_i \). Given any \( \lambda \) and transfer \( x^1 \) to prove Theorem 1, we define another transfer rule \( \tilde{x}^1 \) by

\[
\tilde{x}^1_i(\omega) = x^1_i(\omega) + \rho_i \{ f_i(\omega) + B_i \},
\]

where \( B_i \) is chosen to satisfy

\[
\lambda_i (f_i(\omega) + B_i) \leq 0 \quad \text{for all} \quad \omega \in \Omega,
\]

and \( \rho_i \) is a small positive number. With this construction, \( \sum_{\omega \in \Omega} \lambda_i \tilde{x}^1_i(\omega) \) is less than, or equal to, zero, and is as close to zero as possible as \( \rho_i \to 0 \) for each \( i \).

Now, suppose each player has strict incentive not to deviate at each point \( a(\lambda), \lambda \neq 0 \), under the original sidepayment scheme \( x^1 \). Then, it is obvious that each player would like (i) to conform to \( a(\lambda) \) (because \( a(\lambda) \) is still the unique best reply, provided that \( \rho_i \) is small enough for each \( i \)) and (ii) to tell the truth, under the modified scheme \( \tilde{x}^1 \). Thus we have the following Theorem.

**Theorem 2:** Suppose that there are more than two players \((n > 2)\) and conditions (A2") for \( a = b^j \) (for any \( j \)), (A2), (A3), and (A4) are satisfied. Also suppose that the dimension of \( V^{**} \) is equal to the number of players. Then, any interior point in \( V^{**} \) can be achieved as an average payoff profile of a sequential equilibrium where the players have strict incentives to tell the truth.

**Proof:** Lemma 1 shows that strict incentives for actions can be provided by some scheme \( x^1 \) under the above conditions. (For \( \lambda_j \neq 0 \) and \( \lambda_j = 0, j \neq i \), where player \( i \) is taking a one-shot best response, we could choose \( x^1_i = 0 \) so that player \( i \) has a unique best action.) Then, the theorem is proved as explained above.

**Q.E.D.**

**Remark:** Providing strict incentives for truth-telling for a minimax point in mixed strategies is a little more complicated, but possible. The discussion paper version of the present paper shows that Theorem 2 holds for \( V^{**} \), instead of \( V^{**} \), under an additional assumption that the collection of vectors \( p_{-j}(a_j, \mu_{-j}) \), \( a_j \in A_j \), are linearly independent for all \( i \) and \( j \).

5. **Folk Theorem with Infrequent Communication**

The basic idea in the previous section is to use each player's message to transfer payoffs among others. This induces truthful revelation of private signals, and when each player's signal can discriminate different players' deviations, it provides an efficient enforcement of actions. However, there are some important cases where this idea cannot be applied. Bilateral relationship \((n = 2)\) is a prominent example, which is pervasive in economic applications. Also we noted in Subsection 4.1 that different players' deviations may not be statistically discriminated when the signal space is not rich enough.
In such a situation, we will show that a folk theorem can sometimes be achieved by delaying the release of information, provided that the players' private signals are independently distributed given any pure action profile. The possibility of improving efficiency by the delay of information release has been explored by Abreu, Milgrom, and Pearce (1991) in a model with a publicly observable signal. In our model, their method is not directly applicable, and we will use a different statistical test to prove the folk theorem.9

We consider the following stage game $G^{PD}: n = 2, A_i = \{c, d\}, \Omega_i = \{0, 1\}$, and the expected state payoffs are given by the following table, where $H, L > 0, 1 > H - L$ (the prisoners' dilemma game):

<table>
<thead>
<tr>
<th></th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$1, 1$</td>
<td>$-L, 1 + H$</td>
</tr>
<tr>
<td>$d$</td>
<td>$1 + H, -L$</td>
<td>$0, 0$</td>
</tr>
</tbody>
</table>

We assume that the signals $(\omega_1, \omega_2)$ are independent given any pure action profile. The marginal distributions for $\omega_1$ and $\omega_2$ are symmetric, and $p_1(\omega_1 | a_1, a_2)$ satisfies $p_1(1 | d, d) > p_1(1 | d, c)$ and $p_1(1 | c, d) > p_1(1 | c, c)$. Note that signal 1 is a bad sign which is more likely when the opponent takes $d$. We also specify the message spaces as $M_i = \{\text{Pass}, \text{Fail}\}$, $i = 1, 2$.

A possible interpretation of $G^{PD}$ is the exchange of commodities with uncertain qualities. Player 1's effort level $a_i = c, d$ determines the quality $\omega_2$ of the good she provides to player 2. Here, $\omega_2 = 1$ and $\omega_2 = 0$ mean low and high quality respectively.10 The symmetric explanation applies to player 2's commodity. Note well that the independence assumption, although it is nongeneric, is fairly plausible in this interpretation, as long as the players use different inputs to produce their goods.

**Theorem 3:** When $G^{PD}$ is infinitely repeatedly played with communication, any feasible and individually rational payoff profile can be approximately achieved by $T$-public perfect equilibria, where private information is revealed every $T$ periods, as $\delta \to 1$ and $T \to \infty$.

**Proof:** We will use Proposition 1 and examine Problem $(T, \lambda, \delta^0)$. First we consider the case where $\lambda_1, \lambda_2 > 0$, and $a = (c, c)$ maximizes $\lambda g(a)$. We let the sidepayment $x_i^T$ depend on the last message the opponent sends (i.e., $x_i^T = x_i^T(m_i(T))$ for $i \neq j$), with $x_i^T(\text{Pass}) = 0$, and $x_i^T(\text{Fail}) = -(H + \epsilon)$ for $\epsilon > 0$. We will construct a strategy profile $\tilde{\delta}^T$ in the $T$-time repeated game with sidepayments $(G^{PD})^{T}(\delta, x^T)$ which satisfies (SA), (NOD), and $(\delta^0 \text{-UIC})$ for large

9 Detailed discussion of this point can be found in the discussion paper version of this paper. A recent paper by Compte (1998) employs a different method to prove the folk theorem.

10 If we take this interpretation, the quality of player 1's commodity is affected by her own effort, so we may assume $p_1(\omega_2 | a) = p_1(\omega_2 | a)$, although the following analysis does not use this property.
enough $T$ and also has an efficiency property (EFF): $u_i^T(\delta^T, x^T) \to 1 = g_i(c, c)$ as $T \to \infty$ ($i = 1, 2$).

Strategy $\hat{s}_i^T = (\hat{\sigma}_i^T, \hat{\eta}_i^T)$ is specified as follows. The message strategy $\hat{\eta}_i^T$ for player $i$ always sends "Pass" and "Fail" with equal probability for $t = 1, 2, \ldots, T - 1$, and in period $T$ it sends "Pass" if and only if

$$\frac{1}{T} \sum_{t=1}^T \omega_i(t) \leq p_i(1|c, c) + \xi,$$

where $\xi$ is a positive small number satisfying $\xi < p_i(1|c, d) - p_i(1|c, c) = p_1(1|d, c) - p_2(1|c, c)$. Next we let the action plan $\hat{\sigma}_i^T$ be any strategy which plays $c$ on the equilibrium path for each $t = 1, \ldots, T$.

This clearly satisfies (SA) and (NOD). (EFF) is also satisfied because, by the law of large numbers, the left-hand side of (5.1) almost surely converges to $p_i(1|c, c)$ and there is almost surely no penalty $(x_i^T = 0)$ as $T \to \infty$ on the equilibrium path.

It remains to check condition ($\delta^0$-UIC). Since player $i$'s message does not affect her own payoff, $\hat{\eta}_i^T$ is one of the best responses for player $i$. Given the opponent's message rule and the independence of signals, each player accumulates no information during the $T$ periods. So player $i$'s problem effectively reduces to a (static) choice of action sequence $(a_i(1), \ldots, a_i(T))$. We will show that $(c, \ldots, c)$ is better than any other action sequence.

Formally, let $f_i^T(h)$ be the probability that player $i$ is penalized when player $j$ always cooperates but player $i$ deviates $h$ times. By symmetry it suffices to check player 1's incentives. Strategy $\hat{s}_1^T$ is a best response to $\hat{s}_2^T$ if, for every $h = 1, \ldots, T$, and every action sequence $a_1^T$ which deviates $h$ times, we have

$$1 - (h + \epsilon)f_i^T(0) > \frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^T g_i(a_i(t), c) \delta^{t-1} - (H + \epsilon)f_i^T(h).$$

Since the first term on the right-hand side tends to $1 + hH/T$ as $\delta \to 1$, the above inequality holds for all large enough $\delta$ (implying ($\delta^0$-UIC)) if

$$H + \epsilon)(f_i^T(h) - f_i^T(0)) > hH/T.$$

We will prove that (5.2) holds for all $h = 1, \ldots, T$ by using the main theorem in Matsushima (1995).\footnote{An intuition may be obtained by the following observation. If a player repeatedly deviates (i.e., if $h$ is large), it is easily detected and punished. Note that a huge penalty is required to offset the gains from repeated deviations. On the other hand, a small number of deviations are difficult to detect, but we can utilize the huge punishment, which is designed to deter a large number of deviations.} Matsushima considers the infinitely repeated game where in each stage players simultaneously choose actions in $T$ markets. The game in each market is the prisoner's dilemma with publicly observable signal $\omega = 0, 1$, where $1 > \Pr(\omega = 1|c, d) > \Pr(\omega = 1|c, c) > \Pr(\omega = 1|d, c) > 0$. The
incentive constraint in this game (inequality (14) in Matsushima (1995)) is expressed as

\[(1 - \delta)(T + hH)/T + \delta f^T(h) < (1 - \delta) + \delta f^T(0)\]

for \(h = 1, \ldots, T\), and Matsushima proves that this holds when \(\delta > H/(1 + H)\). Replacing \(\delta\) by \((H + \varepsilon)/(1 + H)\), we have inequality (5.2).

Therefore, in this case the optimal value for Problem \((T, \lambda, \delta^0)\) tends to \(\lambda_1 g_1(c, c) + \lambda_2 g_2(c, c) = \lambda_1 + \lambda_2\) as \(T \to \infty\), and the associated half space \(D(T, \lambda, \delta^0)\) is approximated by \(D\) in Figure 4.

Next we consider the case where \(\lambda_1 > 0, \lambda_2 > 0,\) and \(a = (d, c)\) maximizes \(\lambda g(a)\). We let \(x_1^T = 0, x_2^T(\text{Fail}) = -(L + \varepsilon)(\varepsilon > 0)\), and \(x_2^T(\text{Pass}) = 0\). The strategies in this case are specified as follows. Player 1 always chooses \(d\) while player 1 always takes \(c\). Player 2 always sends each message with equal probability. Player 1 sends each message with equal probability for \(t = 1, \ldots, T - 1\), and in period \(T\) she announces "Pass" if and only if \(\sum_{t=1}^T \omega(t)/T \leq p_1(1/d, d) + \xi'\) for \(0 < \xi' < p_1(1/d, d) - p_1(1/d, c)\). Note that player 1 always takes myopic best
response $d$, while player 2's response is subject to the same statistical test considered above. By the same argument, the half space $D(T, \lambda, \delta^0)$ for this case is approximated by $D'$ in Figure 4.

The case $\lambda_1 \geq 0$ and $\lambda_2 < 0$ is the same as above except that we set $x_1^T(\text{pass}) = L + \epsilon (\epsilon > 0)$ and $x_1^T(\text{Fail}) = 0$ to satisfy $\lambda x^T \leq 0$. The optimal value for Problem $(T, \lambda, \delta^0)$ tends to $\lambda_1 g_1(d, c) + \lambda_2 g_2(d, c) + L = \lambda_1 (1 + H)$ as $T \to \infty$ and $\epsilon \to 0$, and the associated half space $D(T, \lambda, \delta^0)$ is approximated by $D''$ in Figure 4.

The case $\lambda_1, \lambda_2 < 0$ is trivial ($x_1^T = 0$ induces payoff $(0, 0)$), and symmetric arguments apply for the remaining cases. Hence $Q(T, \delta^0)$, the intersection of the half spaces, approximates the individually rational payoff set as $T \to \infty$ (see Figure 4).

Q.E.D.

REMARK: As the proof indicates, providing low equilibrium payoffs is more difficult when we use infrequent communication, and for some games only a subset of the feasible and individually rational payoffs can be achieved by such a class of strategies. In the discussion paper version (Randori and Matsushima (1994)), we have shown the following general result. First, we say that an action profile $a \in A$ is an element of $A'$ if and only if for every $i \in N$, either $g_i(a) \geq g_i(a_i, a')$ for all $a_i \in A_i$ or $\forall a_i, \exists \omega_i \Pr(\omega_i \mid a) = \Pr(\omega_i \mid a_i, \alpha_i)$. For every subset $N'$ of $N$ and for every $a \in A'$, we define $v(N', a)$ by

$$v_i(N', a) = g_i(a) \quad \text{for all } i \not\in N' , \quad \text{and}$$

$$v_i(N', a) = \max_{a_i \in A_i} g_i(a_{-i}, a_i') \quad \text{for all } i \in N'.$$

With $V(N') = \co(v(N', a) \mid a \in A')$, we define

$$V' = \bigcap_{N' \subset N} V(N').$$

Then we can show that any point in $V'$ can approximately be achieved as an equilibrium payoff as $\delta \to 1$, provided that the signals are independent given any (pure) action profile and that the dimension of $V'$ is equal to $n$.

6. CONCLUDING REMARKS

We have shown in the present paper that communication is an important means to resolve possible confusion among players in the course of collusion during repeated play. Confusion may arise when each player observes a different set of signals about other player's past actions. This class of games, known as repeated games with imperfect monitoring and with privately observed signals, includes many important economic applications, such as secret price cutting and exchange of commodities with uncertain quality. The characterization of equilibria in this class of games has been an open question, because the games lack recursive structure and are hard to analyze. We introduced communication to
generate publicly observable history, which recovers the recursive structure. We showed that we can construct equilibria in which the players' private information is voluntarily revealed and is utilized to enforce desirable actions.

One thing which we did not show is the necessity of communication for a folk theorem in this class of games. As we explained above, we do not know how the equilibrium set looks when there is no communication. In principle, there is a possibility that a folk theorem holds even without communication. When we regard a folk theorem as a theory of self-help or cooperation, this may not be a serious problem, as communication is readily available in many cases. On the other hand, if we regard it as a theory of cartel enforcement, it is very important to determine what is possible without communication. This is because certain kinds of communication are *per se* illegal in the antitrust law. If we could show that full collusion is impossible without communication, we would be able to provide a clear-cut theoretical basis for the antitrust law. Thus the characterization of the set of equilibria without communication is a theoretically challenging and economically important open question.\(^\text{12}\)

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*Manuscript received January, 1994; final revision received September, 1996.*

**APPENDIX**

**Proof of Proposition 1:** To prove this proposition, we first show the following lemma.

**Lemma 2:** If a Nash equilibrium \( s' \) in \( \Gamma(G, \bar{v}) \) has no observable deviation, there is a sequential equilibrium which generates the same outcome distribution as \( s' \).

**Proof:** First note that \( s' \) having no observable deviation has the following two implications. First, beliefs on unreached information sets can be derived by Bayes' rule (assuming perfect recall). Hence we can (uniquely) define a system of beliefs \( \pi \) consistent with \( s' \). Secondly, if another strategy profile \( s'' \) is derived from \( s \) by changing actions on unreached information sets, \( \pi \) is also consistent with \( s'' \).

Now let \((s', \pi)\) be defined as above, and modify each player \( i \)'s strategy as follows. Given \( \pi_i \) and some information set for \( i \), a continuation strategy is *optimal* if it achieves the highest continuation payoff. Note that given \( \pi_i \), an optimal continuation strategy exists for any information set by the standard argument (that is, the set of continuation strategies is compact and the payoff function is continuous with respect to the product topology). Now assign an optimal continuation strategy for each unreached information set in the first period, and then do the same for each unreached information set in the second period, and so on. The resulting strategy \( s'' \) is (i) sequentially rational

\(^{12}\) Kandori (1991), Bhaskar and van Damme (1997), and Compte (1994) analyze games with private signals without communication. A recent paper by Sekiguchi (1997) shows a strong result that efficiency can be approximately achieved without communication in the prisoners’ dilemma model, if information is almost perfect.
with respect to \( \pi \), and (ii) different from \( x_i \) in the actions on unreached information sets. Property (ii) and the observation made in the first paragraph of the present proof then imply that \((s^*, \pi)\) is consistent. Thus it is a sequential equilibrium, and (ii) implies that it yields the same outcome distribution as \(s\).

\[Q.E.D.\]

**Proof of Proposition 1**: We will modify Theorem 3.1 of Fudenberg and Levine (1994). First, note that when player \( i \) receives an average payoff of \( v_i \), it can be decomposed into the payoff in the first \( T \) periods and the continuation payoff after that, denoted by \( w_i \):

\[
v_i = (1 - \delta^T) \sum_{t=1}^{T-1} u_i(a_i(t), x_i(t)) \delta^t + \delta^T E[w_i(m^T) | s^T],
\]

where \( s^T \) and \( m^T \) respectively represent the strategies and the messages sent in the first \( T \) periods. Now let us rewrite the average continuation payoff as \( w_i(m^T) = v_i + x_i(m^T X 1 - \delta^T) / \delta^T \). Since the total variation of the continuation payoff is

\[(w_i - v_i) \delta^T / (1 - \delta) = x_i(1 - \delta^T) / (1 - \delta),\]

\( x_i \) times \((1 + \delta + \ldots + \delta^{T-1})\) represents the total discounted variation of the continuation payoff. A simple calculation then shows that \( v_i = \pi_i + E[x_i | s^T] \). Then one can see that the mathematical programming problem (3.2) in Fudenberg and Levine (1994), when applied to the case where (i) the stage game itself is \( T \)-time repeated game \( G^T(\delta) \) and (ii) the continuation payoffs depend on the exchanged messages \( m^T \), translates into our program \((T, \lambda, \delta^0)\): the objective function to be maximized is identical, their conditions (a) and (b) become (a part of) our \((\delta^0, \text{UIC})\), and their condition (c) becomes our condition (B). (Note that we are imposing additional requirements such as (SA), (NOD), and the uniformity of incentive constraints.)

The proof of Theorem 3.1 (ii) of Fudenberg and Levine (1994) in our setting shows the following.

For any compact subset \( W \) of \( Q(T, \delta^0) \), there exists \( \delta^* \) such that \( W \) is self-generating with respect to the stage game \( G^T(\delta^0) \) for all \( \delta \in [\delta^*, 1] \). In particular, for each \( \delta^* \in [\delta^*, 1] \) and each \( v \in W \), there is a strategy profile \( s(v, \delta^*) \) which has the following properties.

(i) It induces expected average payoff profile \( \pi \in G^T(\delta^0) \), \( \delta^* \).

(ii) In each stage game, which is the \( T \)-time repeated game, the same action profile is played throughout the \( T \) periods.

(iii) It is a Nash equilibrium in \( G^T(\delta^0) \).

(iv) It has no observable deviation.

In addition, condition \((\delta^0, \text{UIC})\) in the definition of the contract problem \((T, \lambda, \delta^0)\) implies that \( s(v, \delta^*) \) is a Nash equilibrium in \( G^T(\delta^0) \) for any \( \delta \in [\delta^*, 1] \). Furthermore, since the payoff in \( G^T(\delta^0) \) is defined to be the average payoff, property (ii) ensures that \( s(v, \delta^*) \) achieves the same payoff profile \( \pi \in G^T(\delta^0, \delta^*) \) for any \( \delta \).

Now choose \( \delta^* \) large enough so that we have \( \delta^* \geq \delta^0 \) and \( \delta^{T^*} \geq \delta^* \). The above argument shows that for any point \( v \in W \) and any \( \delta \in [\delta^*, 1] \), there is a Nash equilibrium \( s(v, \delta^*) \) in \( G^T(\delta^0, \delta^*) \) which achieves the payoff profile \( \pi \). Since \( G^T(\delta^0, \delta^*) \) is the same as the usual infinitely repeated game \( G(\lambda, \delta) \) and the argument so far shows that any interior point \( v \) of \( Q(T, \delta^0) \) can be sustained, for each \( \delta \geq \delta^* \), in \( G(\lambda, \delta) \) by a Nash equilibrium which has no observable deviation. Then, Lemma 2 shows that for each \( \delta \geq \delta^* \) there is a sequential equilibrium in \( G(\lambda, \delta) \) that achieves \( v \), and this completes the proof.

\[Q.E.D.\]

Recall that \( G^T(\delta^0, \delta^*) \) is the infinitely repeated game with discount factor \( \delta^* \) whose stage game is the \( T \)-time repeated game \( G^T(\delta^0) \). Note that different discount factors are utilized within and between the stage games.
PROOF OF LEMMA 1: First, take the case where \( \lambda_i \) and \( \lambda_j \) have the same sign. By (A2') and (A3), there is a separating hyperplane defined by normal vector \( y \) such that (a) \( qy < p_{-i}(a)y \) for all \( q \in Q_i(a) \) and (b) \( qy \geq p_{-i}(a)y \) for all \( q \in Q_j(a) \) (see Figure 3). For a positive number \( t \), define \( x_i = y \) and \( x_j = -\lambda_i/\lambda_j x_i \) to satisfy the budget balancing condition (i). By making the parameter \( t \) arbitrarily large, we can make \( (p_{-i}(a) - q_h)x_k \) arbitrarily large for a positive number \( k \). Since \( (p_{-i}(a) - q_h)x_k \) corresponds to the left-hand side of (ii), condition (ii) is satisfied.

Secondly, suppose \( \lambda_i \) and \( \lambda_j \) have different signs. By assumption (A2) and the separating hyperplane theorem, there is \( y \) such that \( qy < p_{-i}(a)y \) for all \( q \in Q_i(a) \) and all \( q \in Q_j(a) \) (see Figure 2). The rest of the proof is exactly the same as above.

Q.E.D.

PROOF OF THEOREM 1: We use the algorithm explained in Section 3 for \( T = 1 \). When \( T = 1 \), Problem (T, \( \lambda, \delta^0 \)) does not depend on \( \delta^0 \), and the condition (SA), (NOD), and (\( \delta^0 \)-UIC) are satisfied when (i) players reveal private signals truthfully and (ii) the strategy profile under consideration is a Nash equilibrium. At the end of each period, we assume that players communicate the signals they received (\( M_1 = \Omega \)). In our construction, each player's message does not affect her continuation payoffs (i.e. \( x_i = x_i(m_{-i}) \)), so that she is willing to tell the truth. We now look at a collection of static contract problems for each welfare weight \( \lambda \in \mathbb{R} \setminus \{0\} \). We examine different cases depending on the signs of \( \lambda \).

Case 1: Player \( i \) is minimaximized (\( \lambda_i < 0, \lambda_j = 0 \) for \( j \neq i \)).

(i) Supported action: \( \mu_i^L \).

(ii) \( x_i = 0 \) and player \( i \) takes on-shot best response \( \mu_i^L \) to \( \mu_i^L \).

(iii) \( x_i(\omega_{-i}) \) provides player \( j \neq i \) correct incentives to take (possibly mixed) strategy \( \mu_j^L \). The existence of such a payment scheme \( x_i \) is proved by Ky Fan's Theorem (Fan (1956)), which asserts that a system of linear inequalities \( P \geq d \) has a solution \( x \) iff \( \beta \geq 0 \) and \( \beta P = 0 \) imply \( \beta d \leq 0 \), where \( \geq \) means element-wise weak inequality. Let \( P \) be the matrix formed by row vectors \( \mu_i^L - \mu_j^L, a_j \), \( a_j \in A_j \), and let \( d \) be the column vector whose elements are \( g(\mu_i^L, a_j) - g(\mu_j^L, a_j) \), \( a_j \in A_j \). Then, Ky Fan's Theorem shows that (A1) is equivalent to the existence of payment scheme \( x_i(\omega_{-i}) \) which provides correct incentives to take strategy \( \mu_i^L \) for player \( j \neq i \).

For other welfare weights \( \lambda \neq 0 \), let \( a(\lambda) \in \arg \max_{a \in A} \lambda g(a) \).

Case 2: Player \( i \) is maximized (\( \lambda_i > 0, \lambda_j = 0 \) for \( j \neq i \)).

(i) Supported action: \( a(\lambda) \).

(ii) \( x_i = 0 \) and player \( i \) takes on-shot best response \( a(\lambda) \) to \( a(\lambda) \).

(iii) \( x_i(\omega_{-i}) \) makes \( a(\lambda) \) a best response for player \( j \neq i \). This is possible by (A2*). The formal proof is similar to Lemma 1 and therefore omitted.

Case 3: Otherwise (there are at least two players with nonzero welfare weight).

(i) Supported action: \( a(\lambda) \).

(ii) For any player \( i \) with zero welfare weight: \( x_i(\omega_{-i}) \) makes \( a(\lambda) \) a best response. This is possible by (A2*).

(iii) For each pair of players \( i \neq j \) whose welfare weights are nonzero, construct a pair of incentive schemes as in Lemma 1.

(iv) If more than one payment scheme has been constructed for player \( i \), let us finally define \( x_i \) to be the sum of those schemes. Clearly, \( x_i \) does not depend on \( \omega \), so that telling the truth is a (weak) best response for player \( i \) and \( \sum_i \lambda_i x_i = 0 \) for all \( \omega \). Furthermore, the incentive constraints are maintained, if we make the number \( d \) in Lemma 1 sufficiently large. Thus we conclude that aside from Case 1, the extremal payoffs in the direction of \( \lambda \) can be achieved: \( D(1, \lambda, \delta^0) = \{ v : \lambda v \leq \lambda g(a(\lambda)) \} \) (recall that \( D \) does not depend on \( \delta^0 \) when \( T = 1 \)). For Case 1, where \( \lambda \) has a negative element for \( i \) and zeros elsewhere, \( D(1, \lambda, \delta^0) = \{ v : v_i \geq 0 \} \). Therefore \( \bigcap_{\delta^0} D(1, \lambda, \delta^0) = \{ v \} = V^* \) and the full dimensionality of \( V^* \) proves the theorem, by means of the characterization in Subsection 3.1.

Q.E.D.
REFERENCES


